Kinematics of Cable-Driven Systems

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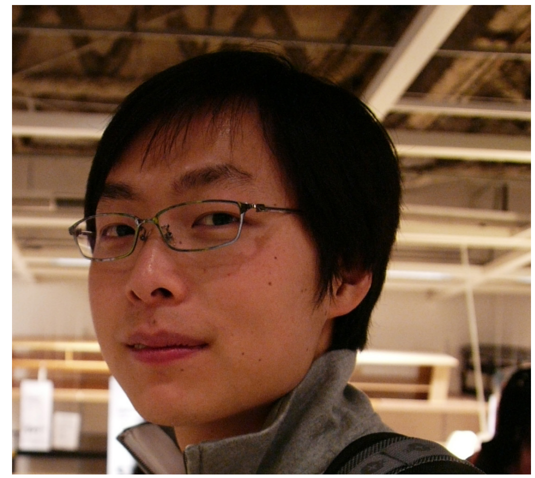
Subhrajit Bhattacharya



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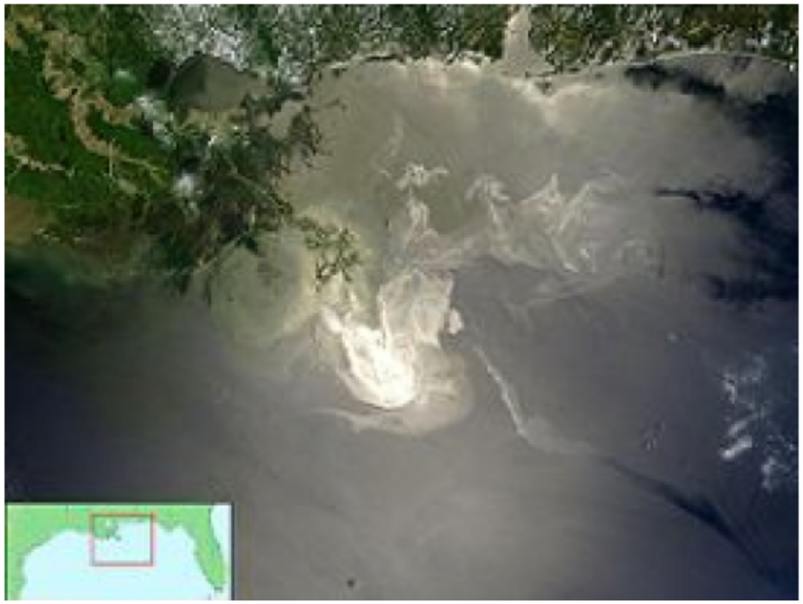
1. Cooperative Skimming



April 20, 2010

Deepwater Horizon drilling rig explosion, Gulf of Mexico

10^4 m^3 of crude oil released into the ocean

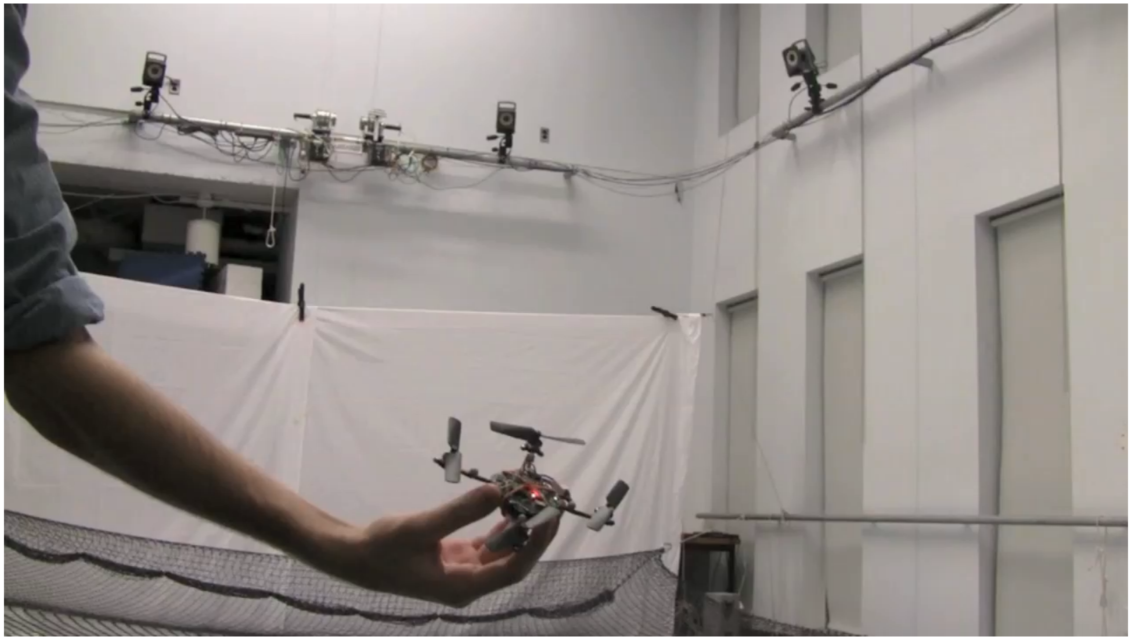
Manual skimming operations at the surface removed ~3% of the oil – **highly inefficient!**

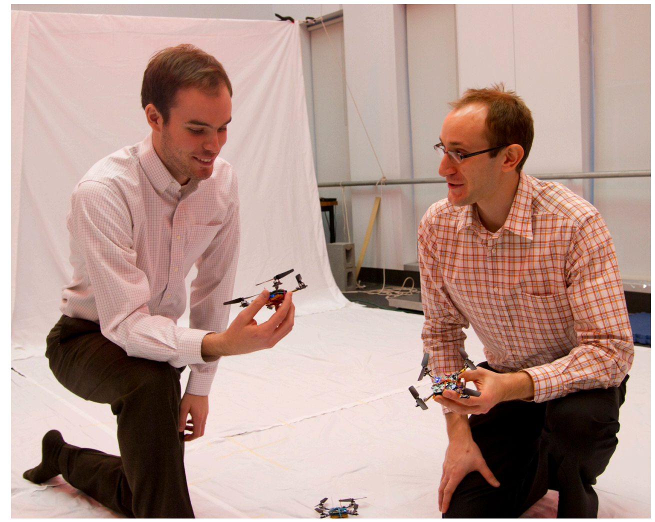
Goal

Develop an efficient **robotic skimming operation** using Autonomous Surface Vehicles

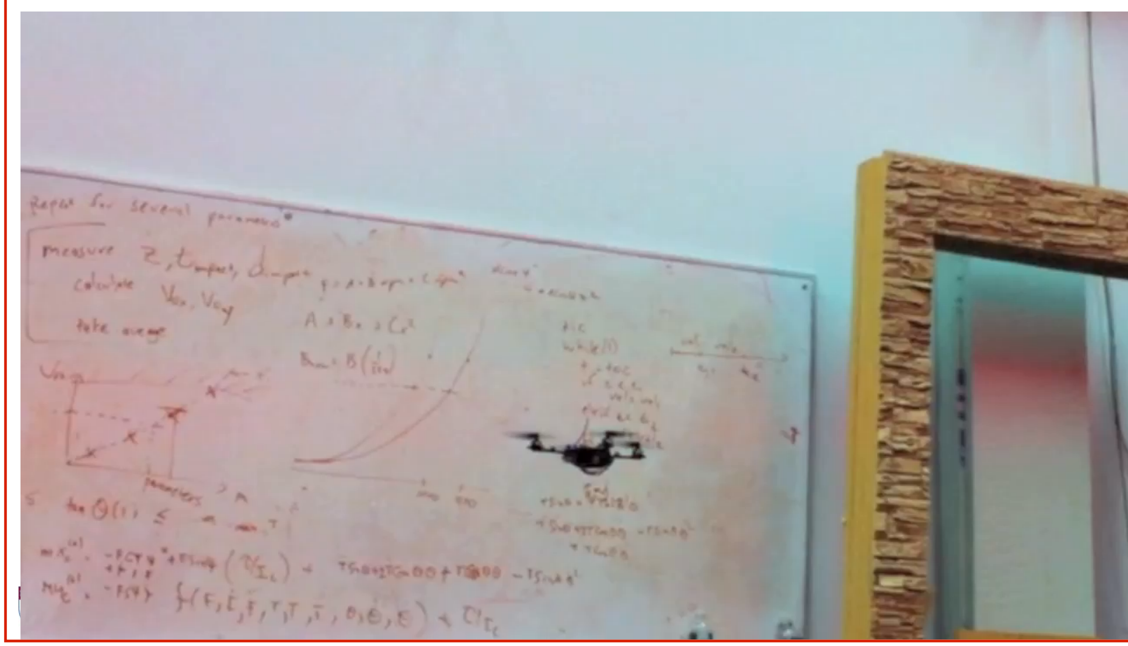
[*Bhattacharya, Heidarsson, Kumar and Sukhatme,* 2011]

Aerial Robots

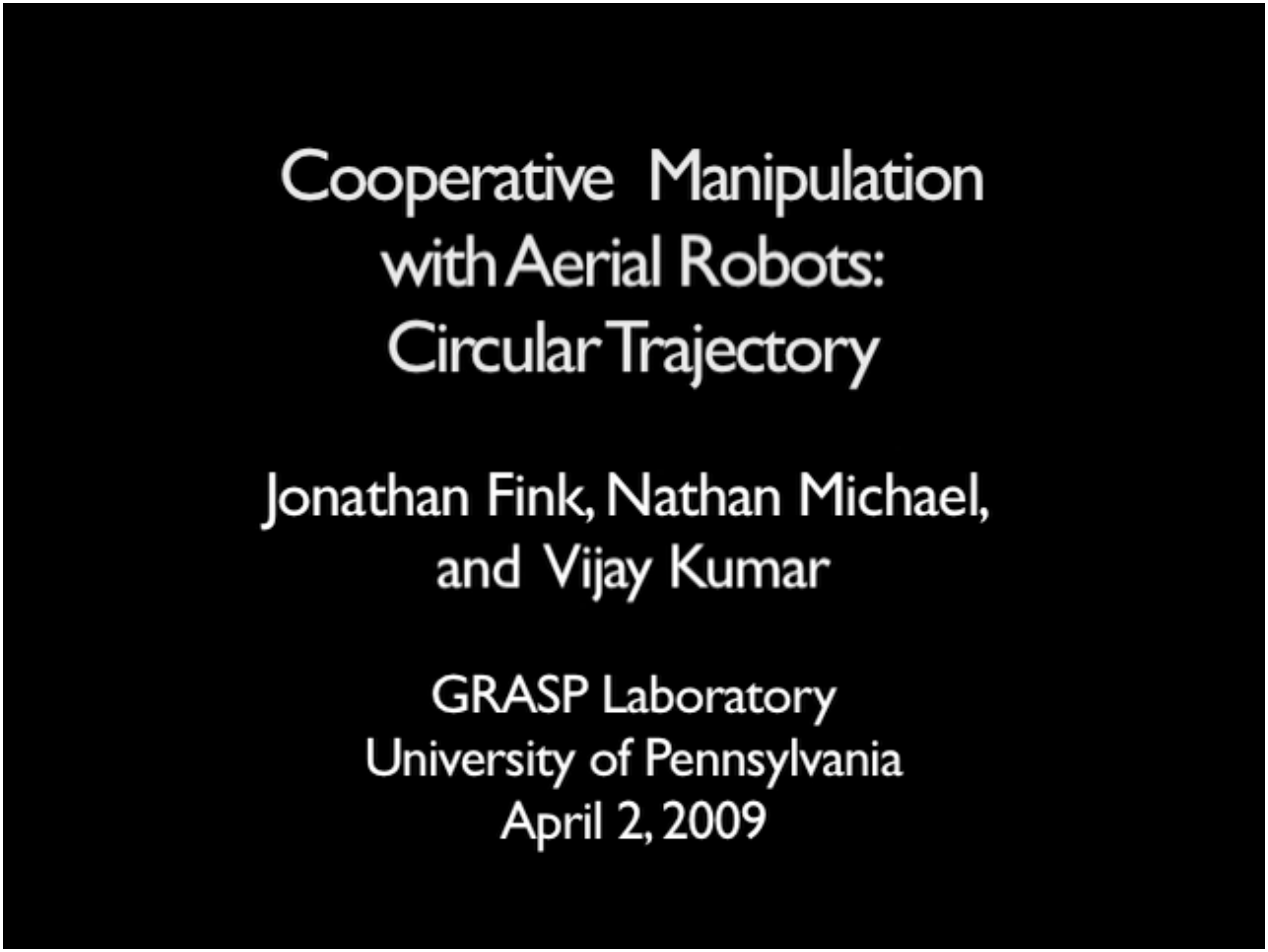




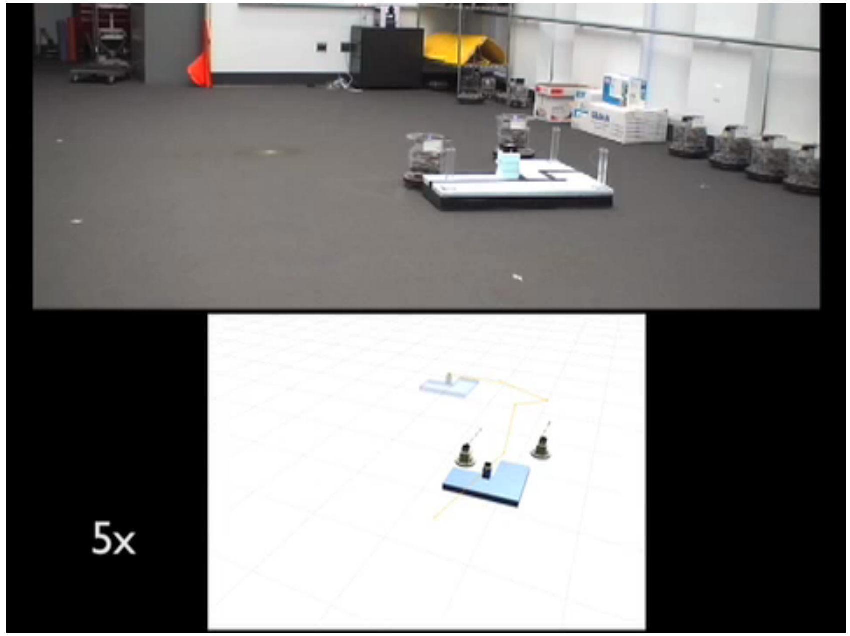
[Kushleyev, Mellinger and Kumar 2012]

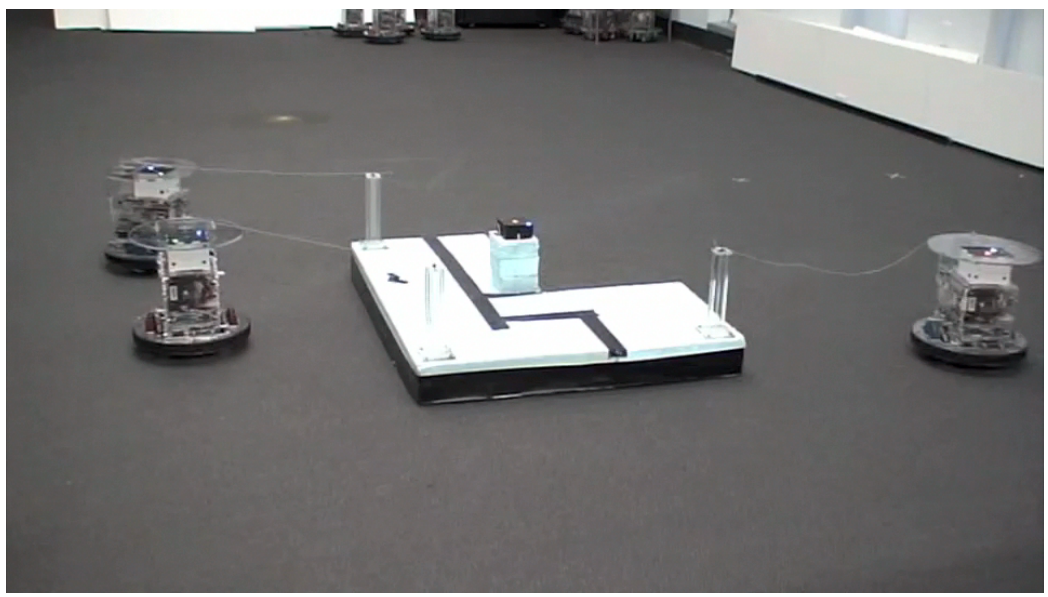


2. Cooperative Manipulation

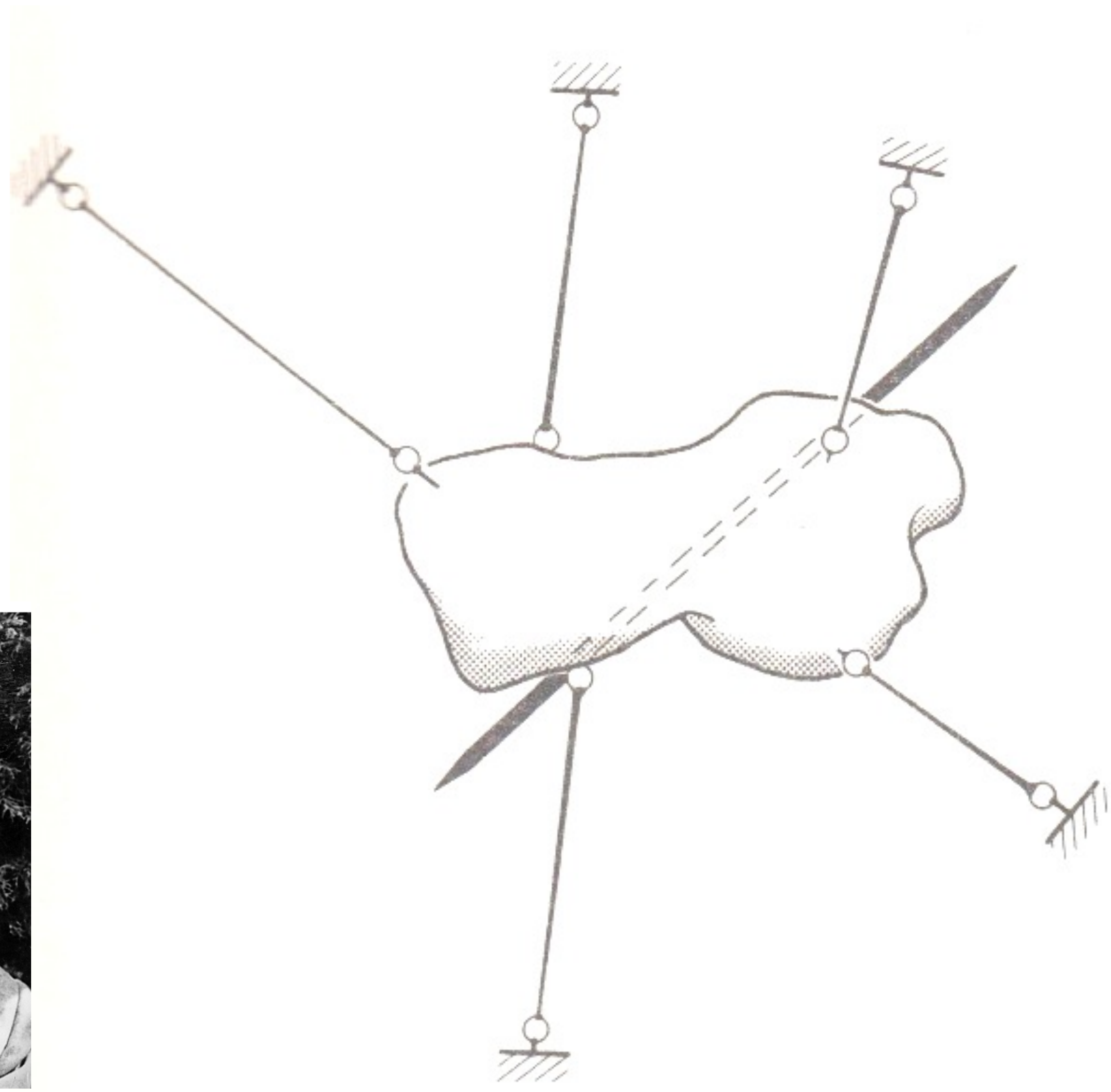


3. Cooperative Towing





Kinematics and Statics of Suspended Payloads



[Möbius, 1837; Ball, 1900]

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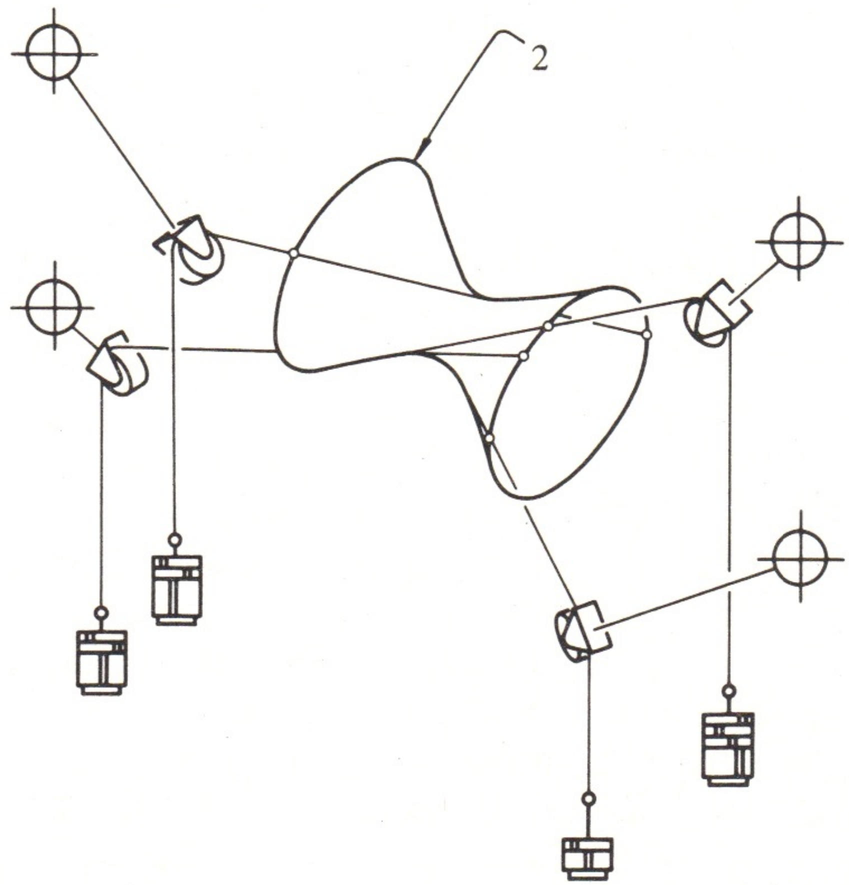
Kinematics and Statics of Suspended Payloads

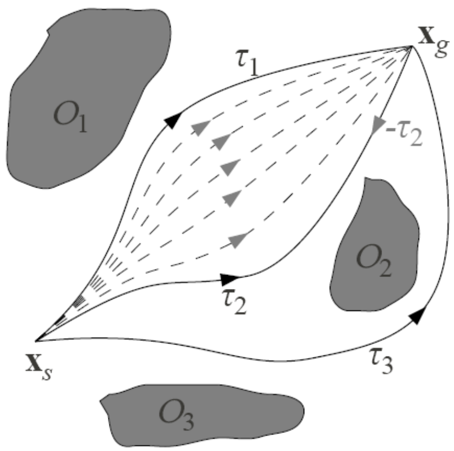


Phillips, J. (1990). Freedom in Machinery, Vol. 1. Cambridge, Cambridge University Press.

Today

1. Direct and inverse kinematics

2. Reasoning about homotopy classes associated with cables and trajectories



The Kinematics of 3-D Cable Towing Systems

ASME IDETC2012 Workshop on 21st Century Kinematics

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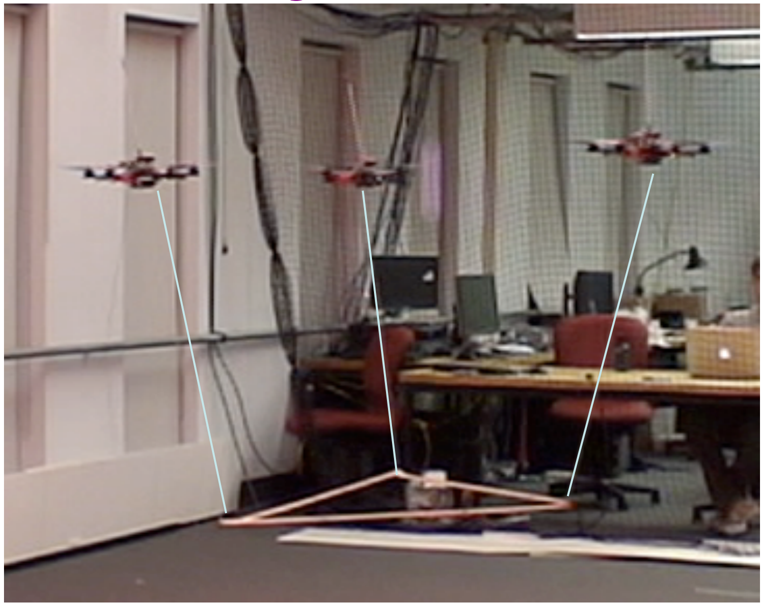
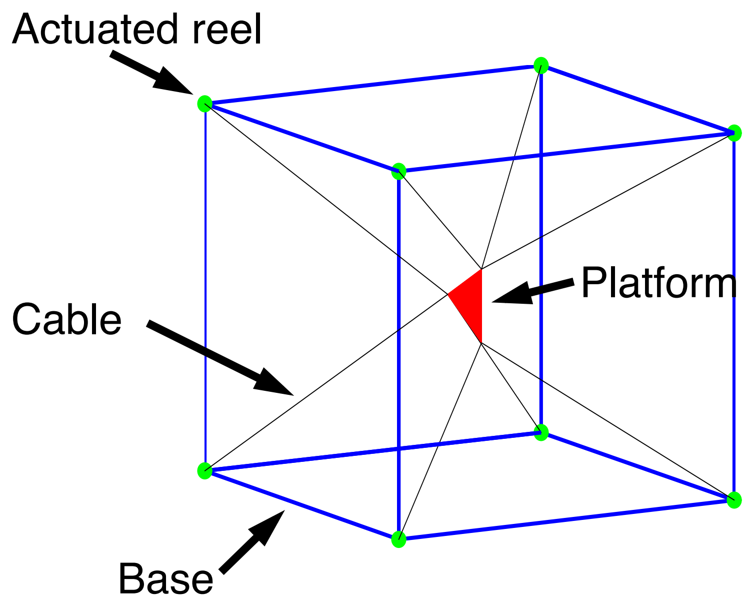
Key Ideas

- Static Equilibrium
- Direct Kinematics
- Inverse Kinematics
- Stability Analysis

Similarity and Difference



3-D Cable Towing System



Cable actuated parallel manipulator

Similarity: Multiple cables are used to control the pose of the payload or platform.

Differences:

Cable lengths	**fixed**	**changing**
Positions of robots or reels	**changing**	**fixed**
Role of weight	**important**	**less important**
workspace	**transport distance**	**inside the frame**
Purpose	**payload transport**	**manipulator**
fundamental	**system of Multiple robots**	**parallel manipulator**

Static Equilibrium Condition

Unit wrench of cable *i* with respect to the origin O of the reference frame:

$$\mathbf{w}_i = \frac{1}{l_i}\begin{bmatrix}\mathbf{q}_i - \mathbf{p}_i \\ \mathbf{p}_i \times \mathbf{q}_i\end{bmatrix}. \quad (1)$$

Wrench caused by the weight of the payload:

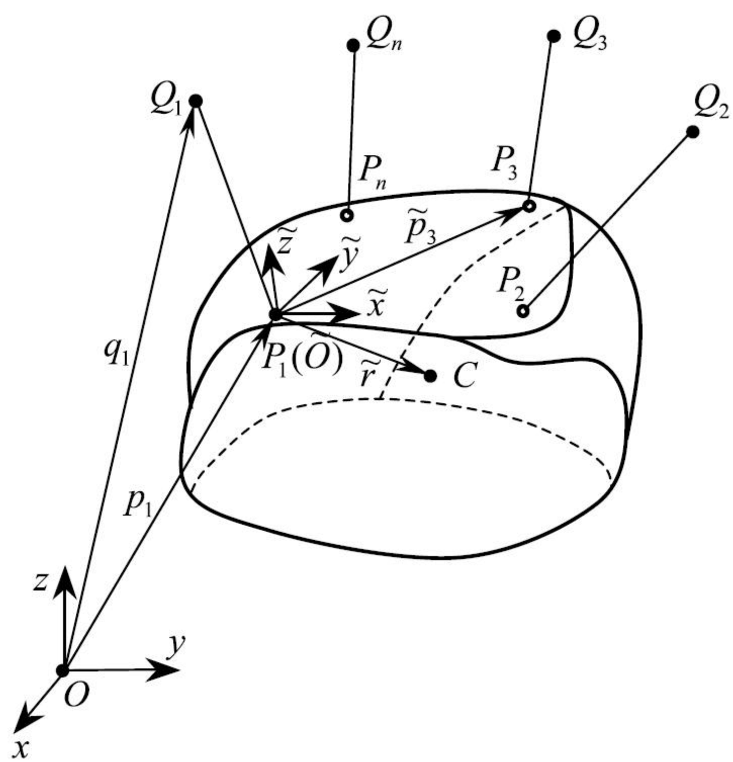
$$\mathbf{G} = -mg\begin{bmatrix}\mathbf{e}_3 \\ \mathbf{r} \times \mathbf{e}_3\end{bmatrix}, \quad (2)$$

Equilibrium equations:

$$[\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \ldots \;\; \mathbf{w}_n]\begin{bmatrix}T_1 \\ T_2 \\ \vdots \\ T_n\end{bmatrix} + \mathbf{G} = 0. \quad (3)$$

Geometric constraints:

$$||\mathbf{q}_i - \mathbf{p}_i|| = l_i. \quad (4)$$



3-D Towing with multiple robots

Direct Kinematics (DK): General case with three robots

Given the positions of the robots, find the possible positions and orientations of the payload that satisfy Eqs.(3) and (4).

P_i can be given as

$$p_i = q_i + \overline{q_i p_i} \quad (i = 1, 2, 3) \qquad (5)$$

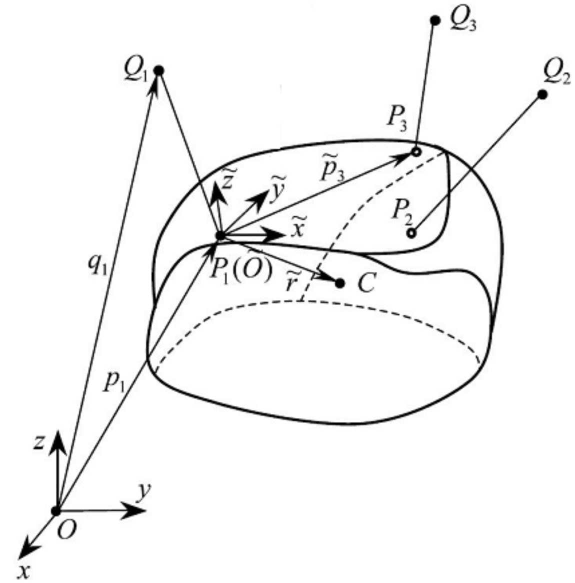
$$\overline{q_i p_i} = [l_i x_i, l_i y_i, l_i z_i]^T$$

$$x_i = \sin\alpha_i \cos\beta_i, \quad y_i = \sin\alpha_i \sin\beta_i, \quad z_i = \cos\alpha_i$$

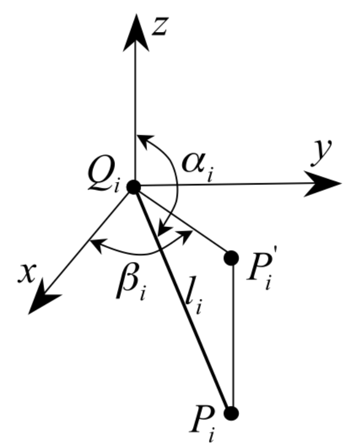
$$x_i^2 + y_i^2 + z_i^2 = 1 \quad (i = 1, 2, 3) \qquad (6)$$

Substituting eqs. (5) and (6) into the equilibrium condition eq.(3):

$$\begin{cases} x_1 T_1 + x_2 T_2 + x_2 T_2 = 0, \\ y_1 T_1 + y_2 T_2 + y_2 T_2 = 0, \\ z_1 T_1 + z_2 T_2 + z_2 T_2 = -mg, \\ (z_{q1} y_1 - y_{q1} z_1) T_1 + (z_{q2} y_2 - y_{q2} z_2) T_2 + (z_{q3} y_3 - y_{q3} z_3) T_3 - mg y_c = 0, \\ (x_{q1} z_1 - z_{q1} x_1) T_1 + (x_{q2} z_2 - z_{q2} x_2) T_2 + (x_{q3} z_3 - z_{q3} x_3) T_3 - mg x_c = 0, \\ (y_{q1} x_1 - x_{q1} y_1) T_1 + (y_{q2} x_2 - x_{q2} y_2) T_2 + (y_{q3} x_3 - x_{q3} y_3) T_3 = 0. \end{cases} \qquad (7)$$



General Case with three robots



Coordinates of P_i

DK: General case with three robots

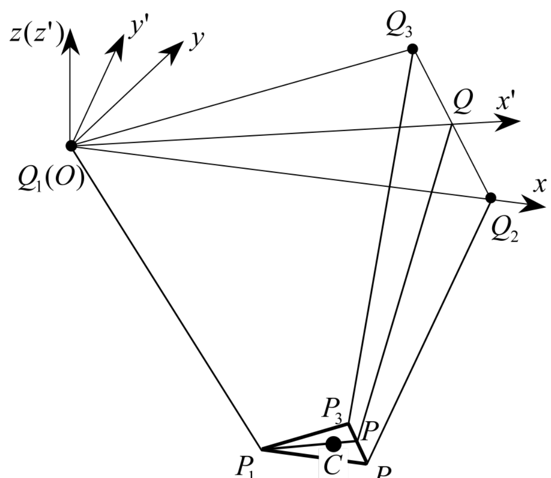
From eq.(7), one gets

$$
\begin{cases}
al_{p2}[(z_{q1}y_1 - y_{q1}z_1)(x_2y_3 - y_2x_3) - (z_{q2}y_2 - y_{q2}z_2)(x_1y_3 - y_1x_3) + (z_{q3}y_3 - y_{q3}z_3)(x_1y_2 - y_1x_2)] \\
\quad + \{al_{p2}(y_{q1} + l_1y_1) + b[l_{p2}(y_{q2} - y_{q1} + l_2y_2 - l_1y_1) + c(y_{q3} - y_{q2} + l_3y_3 - l_2y_2)]\} \\
\quad\quad [x_1(y_2z_3 - y_3z_2) + y_1(x_3z_2 - x_2z_3) + z_1(x_2y_3 - x_3y_2)] = 0, \\
al_{p2}[(x_{q1}z_1 - z_{q1}x_1)(x_2y_3 - y_2x_3) - (x_{q2}z_2 - z_{q2}x_2)(x_1y_3 - y_1x_3) + (x_{q3}z_3 - z_{q3}x_3)(x_1y_2 - y_1x_2)] \\
\quad + \{al_{p2}(x_{q1} + l_1x_1) + b[l_{p2}(x_{q2} - x_{q1} + l_2x_2 - l_1x_1) + c(x_{q3} - x_{q2} + l_3x_3 - l_2x_2)]\} \\
\quad\quad [x_1(y_2z_3 - y_3z_2) + y_1(x_3z_2 - x_2z_3) + z_1(x_2y_3 - x_3y_2)] = 0, \\
(y_{q1}x_1 - x_{q1}y_1)(x_2y_3 - y_2x_3) - (y_{q2}x_2 - x_{q2}y_2)(x_1y_3 - y_1x_3) + (y_{q3}x_3 - x_{q3}y_3)(x_1y_2 - y_1x_2) = 0.
\end{cases}
\tag{8}
$$

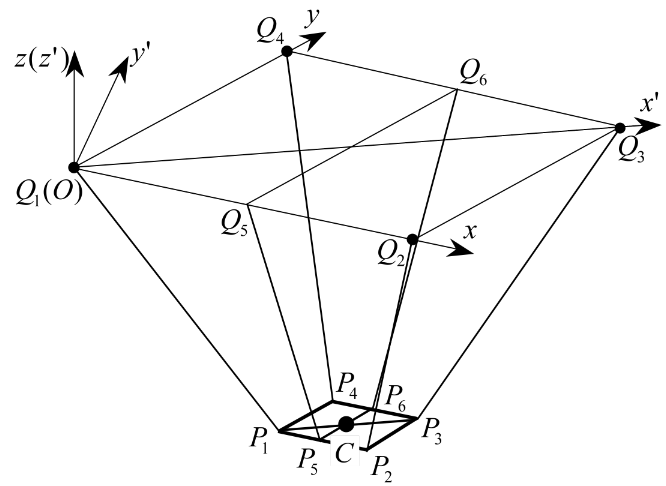
Geometric constraints $\left(\overline{P_1P_2} = l_{p1},\ \overline{P_2P_3} = l_{p2},\ \overline{P_3P_1} = l_{p3}\right)$:

$$
\begin{cases}
l_1[(x_{q1} - x_{q2})x_1 + (y_{q1} - y_{q2})y_1 + (z_{q1} - z_{q2})z_1] - l_1l_2(x_1x_2 + y_1y_2 + z_1z_2) + u_1 \\
\quad + l_2[(x_{q2} - x_{q1})x_2 + (y_{q2} - y_{q1})y_2 + (z_{q2} - z_{q1})z_2] = 0, \\
l_2[(x_{q2} - x_{q3})x_2 + (y_{q2} - y_{q3})y_2 + (z_{q2} - z_{q3})z_2] - l_2l_3(x_2x_3 + y_2y_3 + z_2z_3) + u_2 \\
\quad + l_3[(x_{q3} - x_{q2})x_3 + (y_{q3} - y_{q2})y_3 + (z_{q3} - z_{q2})z_3] = 0, \\
l_1[(x_{q1} - x_{q3})x_1 + (y_{q1} - y_{q3})y_1 + (z_{q1} - z_{q3})z_1] - l_1l_3(x_1x_3 + y_1y_3 + z_1z_3) + u_3 \\
\quad + l_3[(x_{q3} - x_{q1})x_3 + (y_{q3} - y_{q1})y_3 + (z_{q3} - z_{q1})z_3] = 0.
\end{cases}
\tag{9}
$$

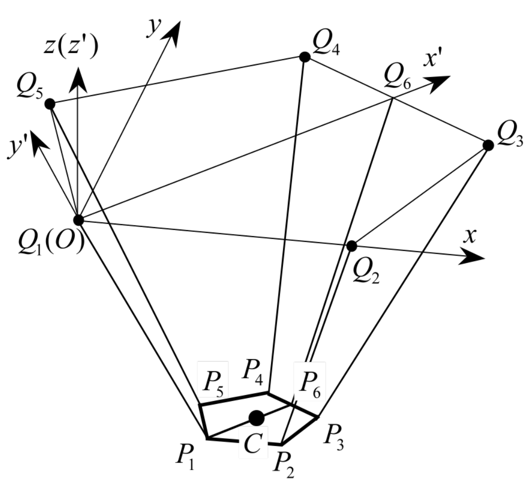
DK: Cable Systems with Symmetric Geometry



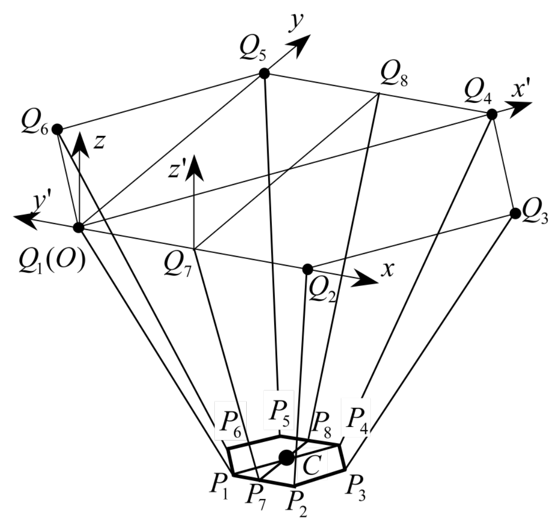
Three cables



Four cables



Five cables



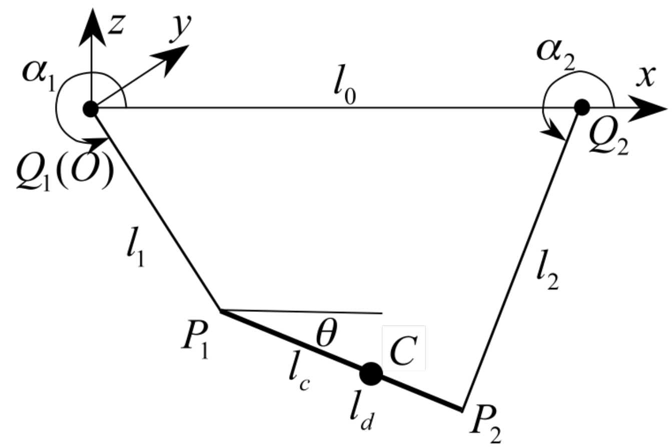
Four cables

DK: Equilibrium problem of planar four-bar linkage

P_i can be given as

$$p_i = q_i + l_i[\cos\alpha_i, \sin\alpha_i] = q_i + l_i[x_i, z_i] \quad (i = 1, 2) \qquad (10)$$

$$x_i^2 + z_i^2 = 1 \quad (i = 1, 2) \qquad (11)$$



Planar four-bar linkage

Equilibrium condition:

$$\begin{cases} x_1 T_1 + x_2 T_2 = 0, \\ z_1 T_1 + z_2 T_2 + mg = 0, \\ l_0 l_d z_2 T_2 + mg[l_1(l_d - l_c)x_1 + l_2 l_c x_2 + l_0 l_c] = 0. \end{cases} \qquad (12)$$

$$[l_1(l_c - l_d)x_1 x_2 - l_2 l_c x_2^2 - l_0 l_c x_2]z_1 - [l_1(l_c - l_d)x_1^2 - l_2 l_c x_1 x_2 + l_0(l_d - l_c)x_1]z_2 = 0 \qquad (13)$$

Geometric constraints $\overline{P_1P_2} = l_d$:

$$l_1 l_2 z_1 z_2 + l_1 l_2 x_1 x_2 + l_0(l_1 x_1 - l_2 x_2) + t_1 = 0 \qquad (14)$$

8th degree polynomial in x_1

$$a_8 x_1^8 + a_7 x_1^7 + a_6 x_1^6 + a_5 x_1^5 + a_4 x_1^4 + a_3 x_1^3 + a_2 x_1^2 + a_1 x_1 + a_0 = 0 \qquad (15)$$

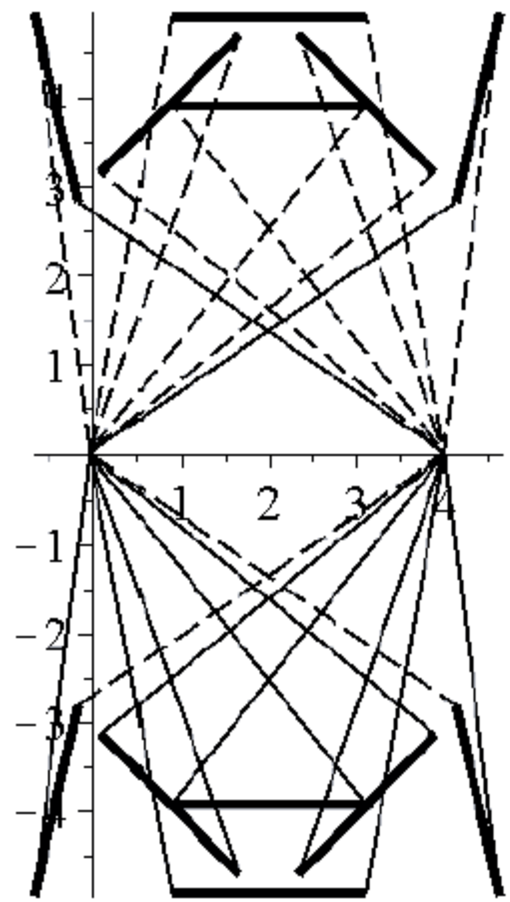
DK: An example of four-bar linkage

Table 1 The used parameters of the planar 4-bar linkage.

$l_0(m)$	$l_1(m)$	$l_2(m)$	$l_d(m)$	$l_c(m)$	$mg(N)$
4	5	5	2.2	1.1	10

Table 2 The solutions of the equilibrium problem of the planar 4-bar linkage.

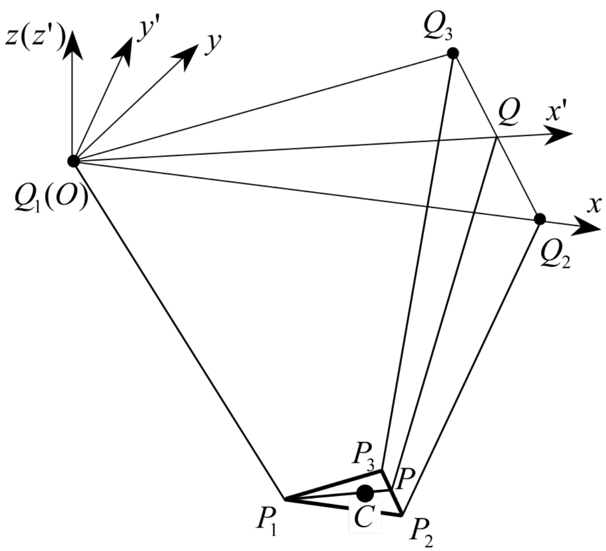
$No.$	x_1	x_2	z_1	z_2
1	0.826	0.127	0.564	0.992
2	0.826	0.127	- 0.564	-0.992
3	0.777	-0.332	0.630	0.943
4	**0.777**	**-0.332**	**- 0.630**	**-0.943**
5	0.620	-0.620	0.785	0.785
6	**0.620**	**-0.620**	**-0.785**	**-0.785**
7	-0.127	-0.826	0.992	0.564
8	-0.127	-0.826	-0.992	-0.564
9	0.332	-0.777	0.943	0.630
10	**0.332**	**-0.777**	**-0.943**	**-0.630**
11	0.180	-0.180	0.984	0.984
12	**0.180**	**-0.180**	**-0.984**	**-0.984**
13	-1	NA	NA	NA
14	-1	NA	NA	NA
15	1	2.116	0	NA
16	1	2.116	0	NA



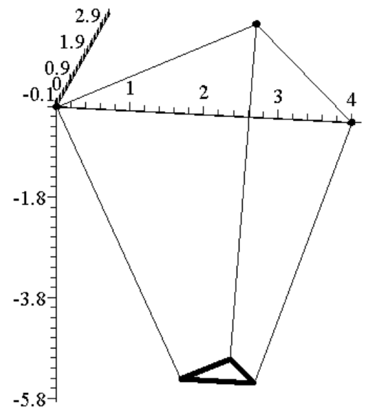
The 12 equilibrium configurations of the planar 4-bar linkage.

DK: Solutions based on planar four-bar linkage
____The case with three robots

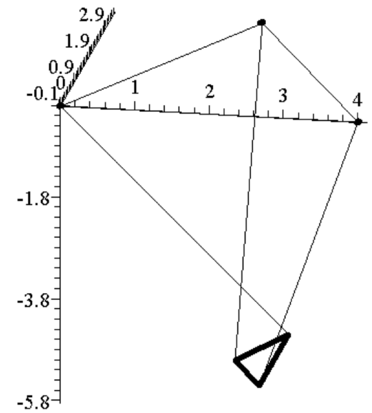
$$l = 6m,\ l_q = 4m,\ l_p = 1m$$



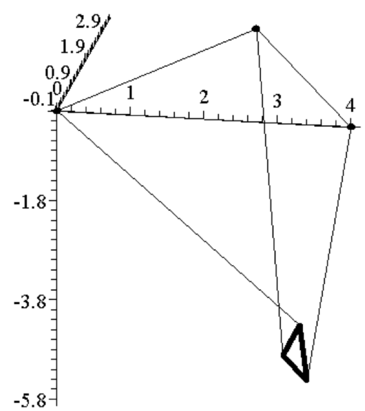
Initial configuration



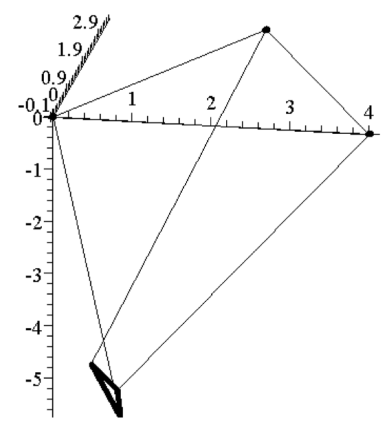
Configuration1 (Stable)



Configuration 2 (Stable)



Configuration 3 (Unstable)

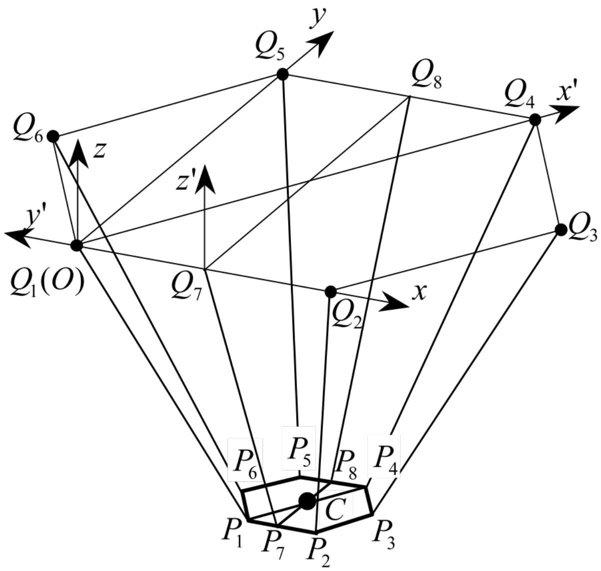


Configuration 4 (Unstable)

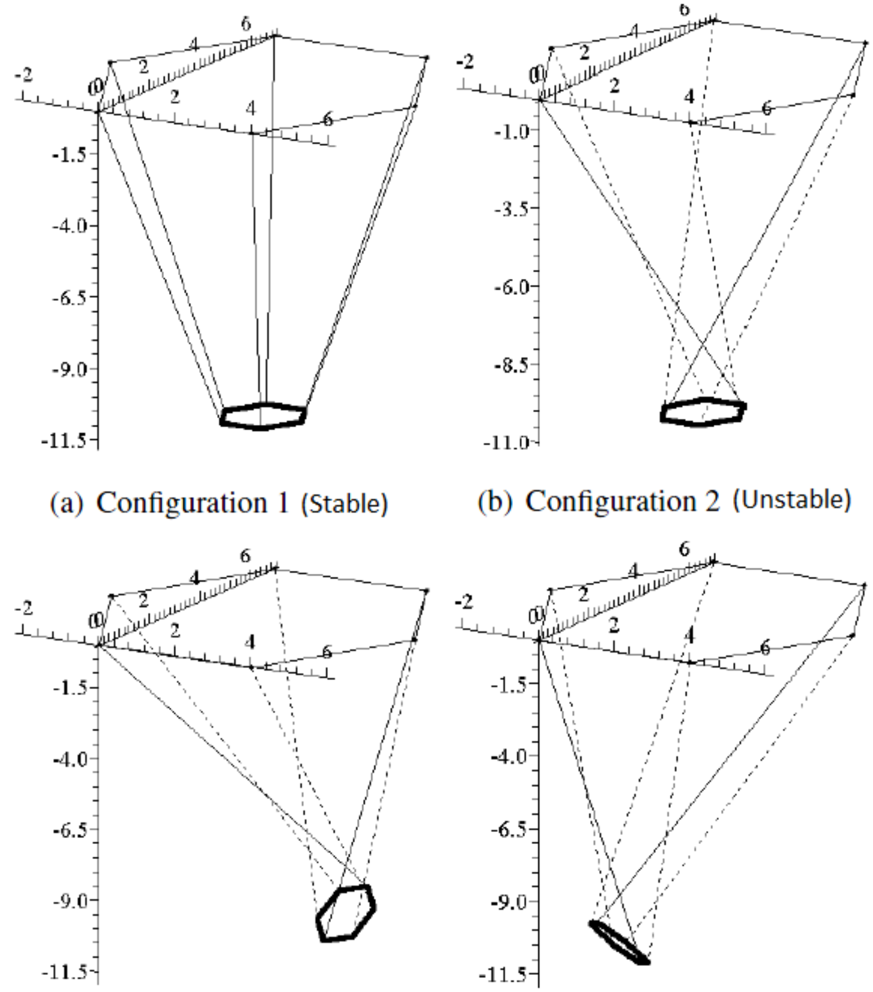
Four equilibrium configurations in plane Q_1P_1PQ

DK: Solutions based on planar four-bar linkage ____The case with six robots

$$l = 12m,\ l_q = 4m,\ l_p = 1m.$$



Initial configuration



(a) Configuration 1 (Stable)

(b) Configuration 2 (Unstable)

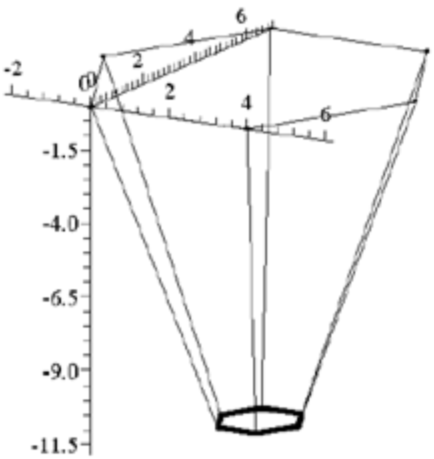
(c) Configuration 3 (Unstable)

(d) Configuration 4 (Unstable)

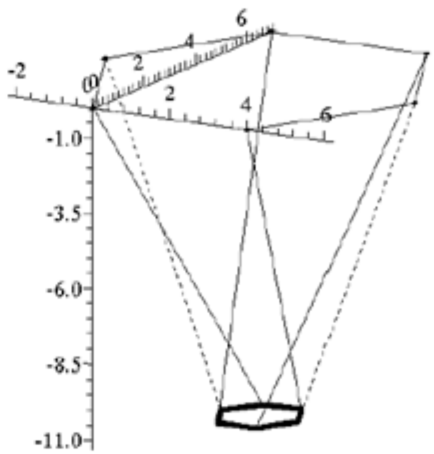
Four equilibrium configurations in the plane $Q_1P_1P_4Q_4$.

DK: Solutions based on planar four-bar linkage

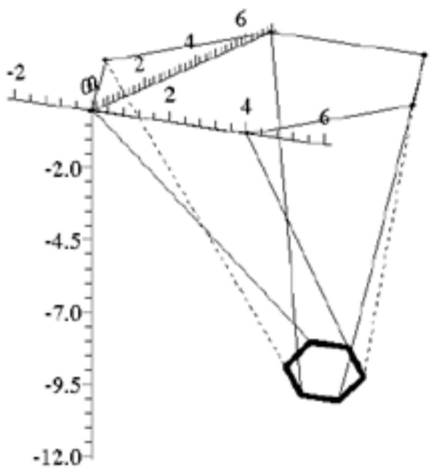
____The case with six robots



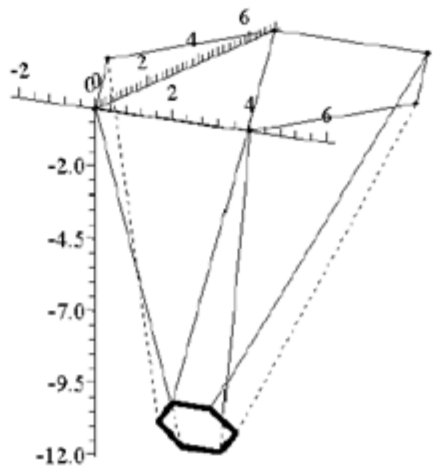
(a) Configuration 5 (Stable)



(b) Configuration 6 (Stable)

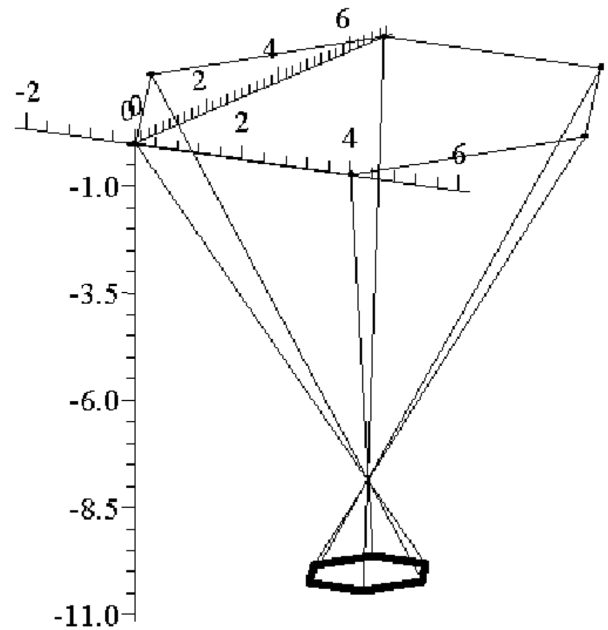


(c) Configuration 7 (Unstable)



(d) Configuration 8 (Unstable)

Four equilibrium configurations in the plane $Q_7P_7P_8Q_8$



Configuration 7 (Unstable)

Inverse Kinematics (IK)

Given the desired position and orientation of the payload, find the positions of the robots that satisfy the equilibrium equations and the geometric constraints.

Assume cable tensions (T_i) are given. From equilibrium equations:

$$\begin{cases} s_1 x_1 + s_2 x_2 + s_3 x_3 = 0, \\ s_1 y_1 + s_2 y_2 + s_3 y_3 = 0, \\ s_1 z_1 + s_2 z_2 + s_3 z_3 = 0, \\ -s_6 y_1 + s_5 z_1 - s_9 y_2 + s_8 z_2 - s_{12} y_3 + s_{11} z_3 = t_1, \\ s_6 x_1 - s_4 z_1 + s_9 x_2 - s_7 z_2 + s_{12} x_3 - s_{10} z_3 = t_2, \\ -s_5 x_1 + s_4 y_1 - s_8 x_2 + s_7 y_2 - s_{11} x_3 + s_{10} y_3 = 0, \end{cases} \tag{16}$$

Constants

where s_1, s_2, …, s_{12}, t_1, t_2 are constants or functions of of T_i (i=1,2,3).

From geometric constraints:

$$x_i^2 + y_i^2 + z_i^2 = l_i^2 \qquad (i = 1, 2, 3) \tag{17}$$

$$x_i = x_{qi} - x_{pi},\ y_i = y_{qi} - y_{pi},\ z_i = z_{qi} - z_{pi}$$

IK(…contd.)

Note equilibrium equations are linearly independent in (z_1, y_2, z_2, x_3, y_3, z_3),

$$\begin{cases} z_1 = t_{17}x_1 + t_{18}y_1 + t_{19}x_2 + t_{20}, \\ y_2 = -(t_4x_1 + t_8y_1 + t_6x_2)/t_9, \\ x_3 = -(s_1x_1 + s_2x_2)/s_3, \\ y_3 = t_{11}x_1 + t_{12}y_1 + t_{13}x_2, \\ z_2 = t_{21}x_1 + t_{22}y_1 + t_{23}x_2 + t_{24}. \end{cases} \tag{18}$$

where coefficients (t_{17}, t_{18}, …, t_{24}) are functions of T_i (i=1,2,3).

Substituting Eq.(18) into Eq.(17), we get **three quadratic** equations:

$$\begin{cases} a_1x_1^2 + b_1y_1^2 + c_1x_2^2 + d_1x_1y_1 + e_1y_1x_2 + f_1x_2x_1 + g_1x_1 + h_1y_1 + i_1x_2 + j_1 = 0, \\ a_2x_1^2 + b_2y_1^2 + c_2x_2^2 + d_2x_1y_1 + e_2y_1x_2 + f_2x_2x_1 + g_2x_1 + h_2y_1 + i_2x_2 + j_2 = 0, \\ a_3x_1^2 + b_3y_1^2 + c_3x_2^2 + d_3x_1y_1 + e_3y_1x_2 + f_3x_2x_1 + g_3x_1 + h_3y_1 + i_3x_2 + j_3 = 0. \end{cases} \tag{19}$$

IK: Analytic algorithm based on Dialytic elimination

Suppressing x_2, we get

$$a_i x_1^2 + b_i y_1^2 + d_i x_1 y_1 + k_i x_1 + u_i y_1 + v_i = 0 \qquad (i = 1,2,3) \tag{20}$$

$$x_1 = X/T,\ y_1 = Y/T \qquad k_i = f_i x_2 + g_i,\ u_i = e_i x_2 + h_i,\ v_i = c_i x_2^2 + i_i x_2 + j_i$$

$$a_i X^2 + b_i Y^2 + d_i XY + k_i XT + u_i T^2 = F_i = 0 \qquad (i = 1,2,3) \tag{21}$$

$$F_{iX} X + F_{iY} Y + F_{iT} T = 0 \qquad (i = 1,2,3) \tag{22}$$

$$F_{iX} = \frac{\partial F_i}{\partial X},\ F_{iY} = \frac{\partial F_i}{\partial Y},\ F_{iT} = \frac{\partial F_i}{\partial T}$$

$$J X_1 = 0 \tag{23}$$

$$J = \begin{bmatrix} F_{1X} & F_{1Y} & F_{1T} \\ F_{2X} & F_{2Y} & F_{2T} \\ F_{3X} & F_{3Y} & F_{3T} \end{bmatrix}, \quad X_1 = [X, Y, T]^T$$

IK: Analytic algorithm based on Dialytic elimination
(Salmon 1885, Roth 1993)

$$JX_1 = 0 \quad (24)$$

$$|J| = 0 \quad (25)$$

Functions of x_2

$$\begin{cases} \dfrac{\partial |J|}{\partial X} = 3AX^2 + 2BXY + 2CXT + DY^2 + ET^2 + FYT = 0, \\ \dfrac{\partial |J|}{\partial Y} = BX^2 + 2DXY + FXT + 3GY^2 + 2HYT + IT^2 = 0, \\ \dfrac{\partial |J|}{\partial T} = CX^2 + 2EXT + FXY + HY^2 + 2IYT + 3JT^2 = 0. \end{cases} \quad (26)$$

From eqs.(21) and (26), we get

$$MX_2 = 0 \quad (27)$$

$M : 6 \times 6$ *matrix.* $X_2 = [X^2, Y^2, XY, XT, YT, T^2]^T$.

$$|M| = f(x_2) = 0 \quad (28)$$

8th degree polynomial in x_2

IK: Case study – equilateral triangle payload

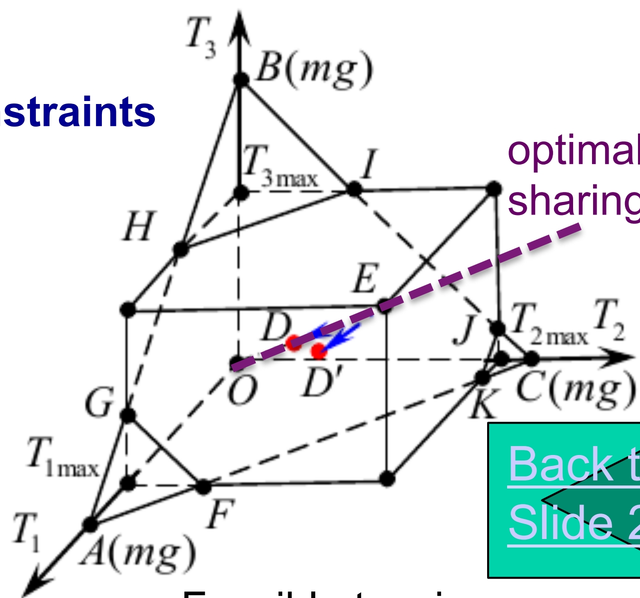
Specify load distribution

1. Normalized load (tension)

$$c_{ri} = T_i / T_{i\max}$$

2. Tension constraints

$$\sum_{i=1}^{3} T_i \geq mg$$



Feasible tensions

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Used parameters

$\tilde{p}_1 = [0, 0, 0]^T$, $\tilde{p}_2 = [1, 0, 0]^T$, $\tilde{p}_3 = [0.5, \sqrt{3}/2, 0]^T$

$\tilde{r} = [0.5, \sqrt{3}/6, 0]^T$, $r = [1, 1, 1]^T$

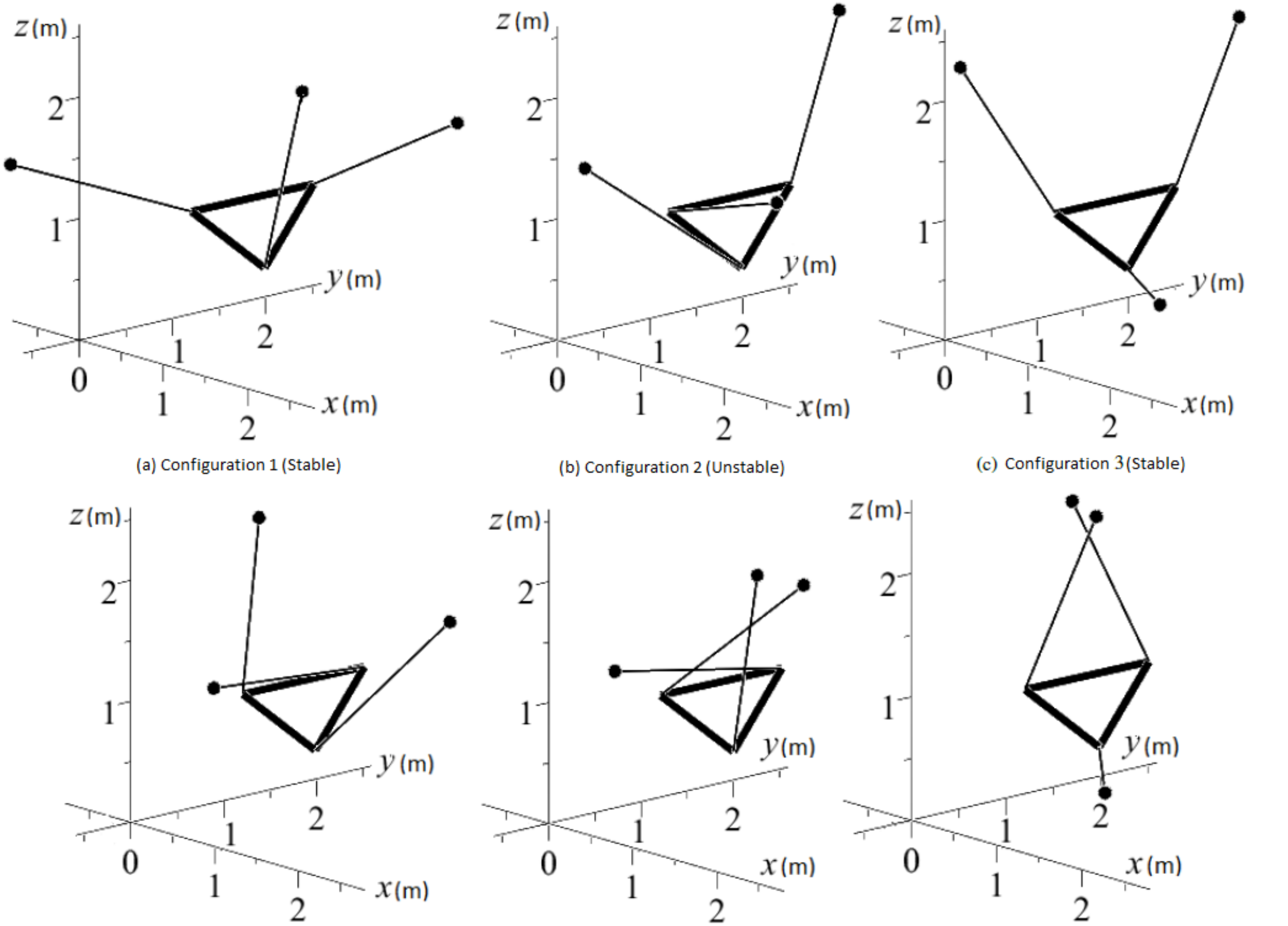
$mg = 25N$, $\lambda_{i\max} = 20N$, $l_i = 1.5m\,(i = 1, 2, 3)$

$\phi = 25^\circ$, $\theta = 15^\circ$, $\psi = -5^\circ$

Table 3: Only 6 real solutions for an equilateral triangle payload with $c_r=0.8$.

No.	x_{q1}	y_{q1}	z_{q1}	x_{q2}	y_{q2}	z_{q2}	x_{q3}	y_{q3}	z_{q3}
1	-0.430	-0.347	1.424	1.045	1.447	1.996	2.385	1.900	1.924
2	1.907	0.646	1.399	0.277	0.052	1.466	0.816	2.302	2.479
3	-0.644	0.743	2.021	2.697	-0.091	0.849	0.946	2.348	2.473
4	-0.072	1.445	2.247	2.105	1.532	1.801	0.967	0.024	1.296
5	1.351	1.526	1.968	0.946	1.395	1.992	0.703	0.079	1.384
6	0.359	1.632	2.244	2.570	-0.271	0.787	0.071	1.638	2.312

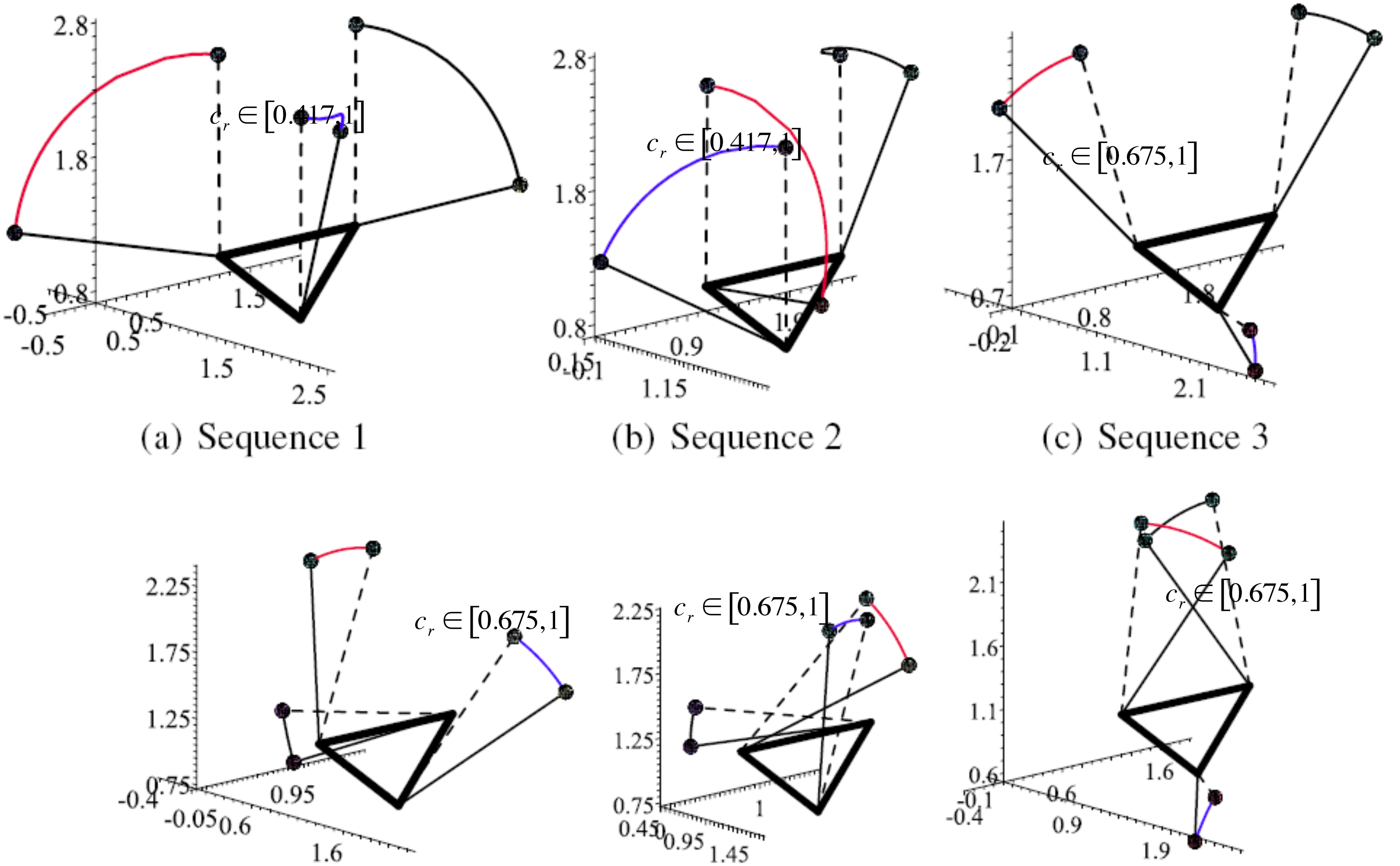
IK: Fixed load distribution ratio



(a) Configuration 1 (Stable)
(b) Configuration 2 (Unstable)
(c) Configuration 3 (Stable)
(d) Configuration 4 (Unstable)
(e) Configuration 5 (Unstable)
(f) Configuration 6 (Stable)

Six configurations for an equilateral triangle payload with c_r=*0.8*.

IK: Effect of changing the load distribution ratio, c_r



(a) Sequence 1 (b) Sequence 2 (c) Sequence 3

(d) Sequence 4 (e) Sequence 5 (f) Sequence 6

The six sequences of configuration for an equilateral triangle payload as c_r is varied. $c_{r,min}$ = 0.417.

IK: General payload

$No.$	x_{q1}	y_{q1}	z_{q1}	x_{q2}	y_{q2}	z_{q2}	x_{q3}	y_{q3}	z_{q3}
1	-0.024	1.473	2.621	2.385	1.453	1.790	0.588	0.071	1.979
2	-0.590	-0.319	1.845	0.650	1.464	2.186	2.531	1.405	2.214
3	1.783	0.527	1.738	0.030	0.085	1.630	1.293	1.977	2.781
4	1.469	1.294	2.196	1.039	1.656	2.195	0.646	0.028	1.944
5	-0.111	1.428	2.618	2.548	-0.166	0.937	0.510	1.522	2.728
6	-0.456	1.159	2.560	2.489	-0.244	0.911	0.821	1.791	2.794

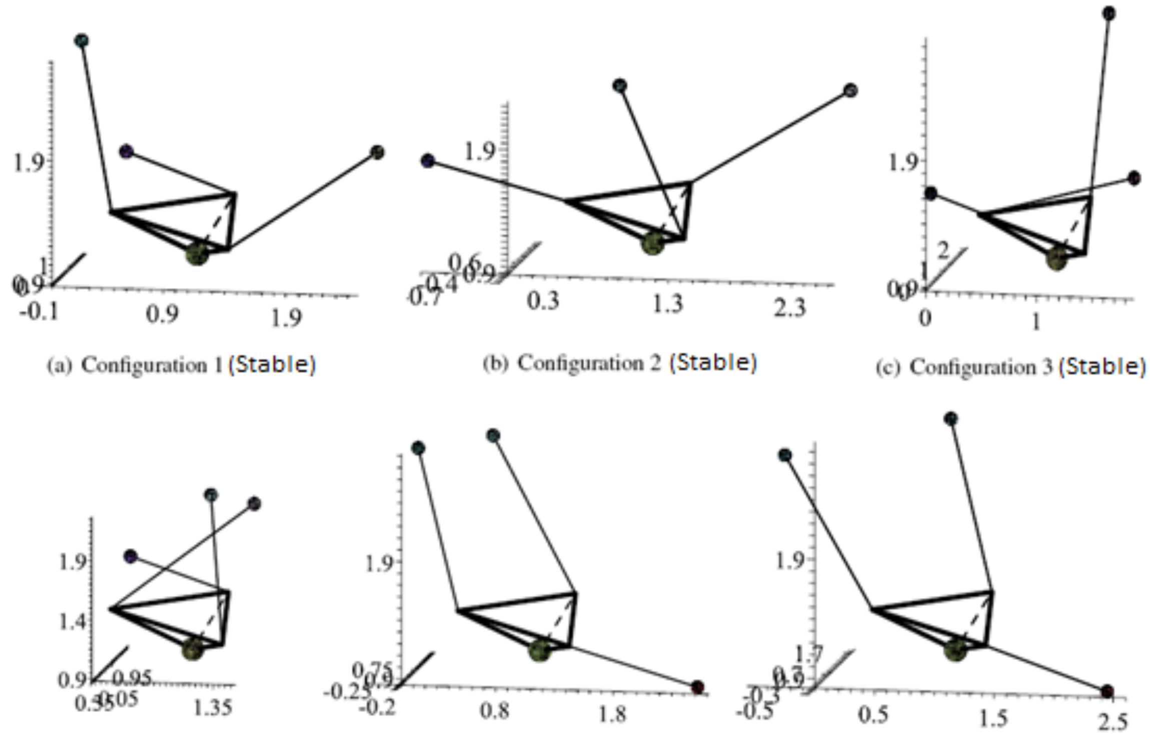
Used parameters

$\tilde{p}_1 = [0, 0, 0]^T$, $\tilde{p}_2 = [1, 0, 0]^T$, $\tilde{p}_3 = [0.8, 0.7, 0]^T$

$\tilde{r} = [0.7, 0.2, -0.3]^T$, $r = [1,\ 1,\ 1]^T$

$mg = 100N$, $\lambda_{1\max} = 60N$, $\lambda_{2\max} = 70N$, $\lambda_{3\max} = 80N$,

$l_i = 1.5m\,(i = 1, 2, 3)$, $\quad \phi = 25^\circ$, $\theta = 15^\circ$, $\psi = -5^\circ$

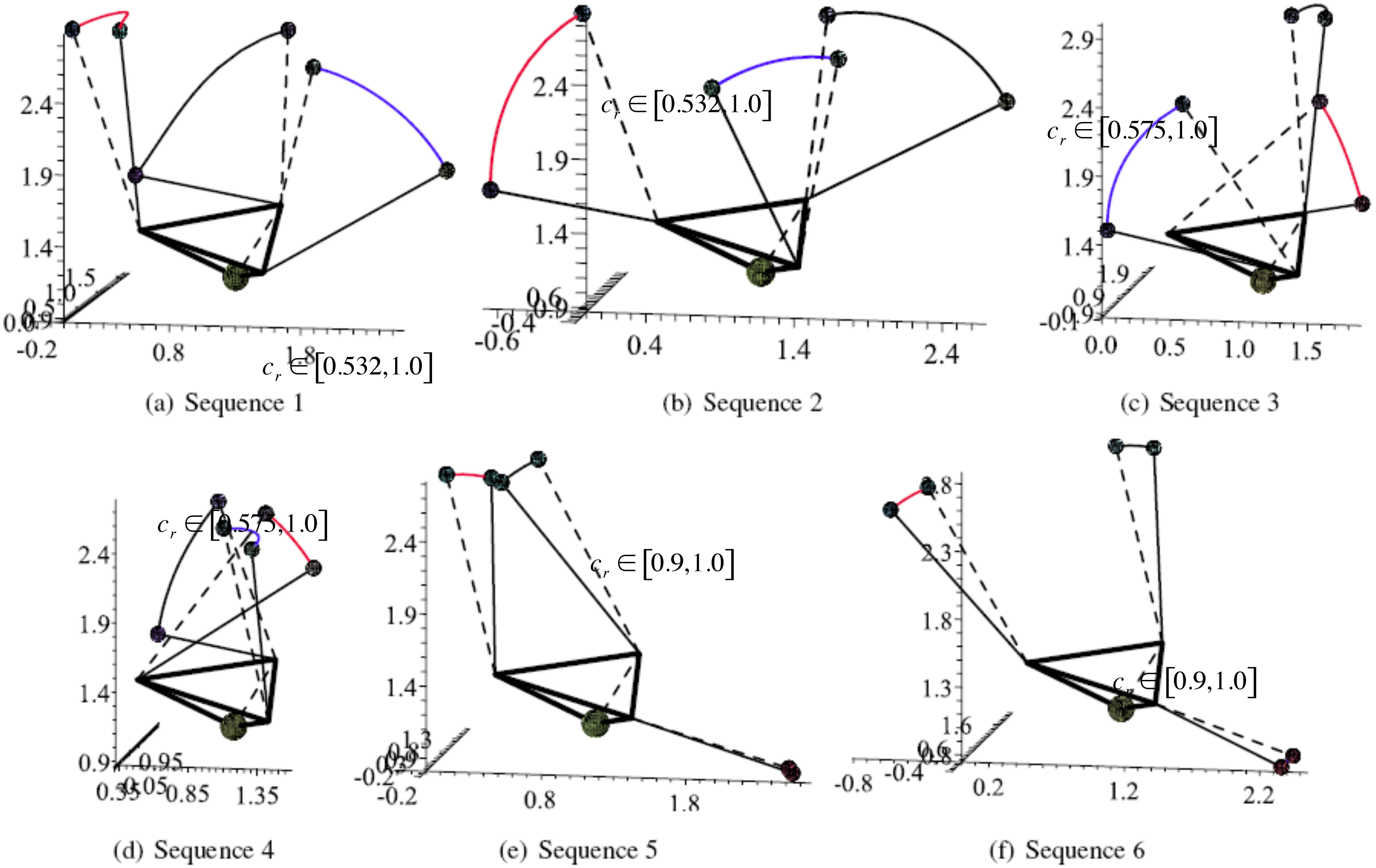


(a) Configuration 1 (Stable) (b) Configuration 2 (Stable) (c) Configuration 3 (Stable)

(d) Configuration 4 (Unstable) (e) Configuration 5 (Stable) (f) Configuration 6 (Stable)

Six configurations for a general payload with c_r=*0.9*.

IK: General payload: Changing the load distribution ratio, c_r



The six sequences of configuration of a general payload (3-D, center of mass not at centroid of triangle of anchor points)

IK: Tension workspace

Definition: The tension workspace can be defined as the sets of tensions at which at least one configuration can be found for a given position and orientation of the payload.

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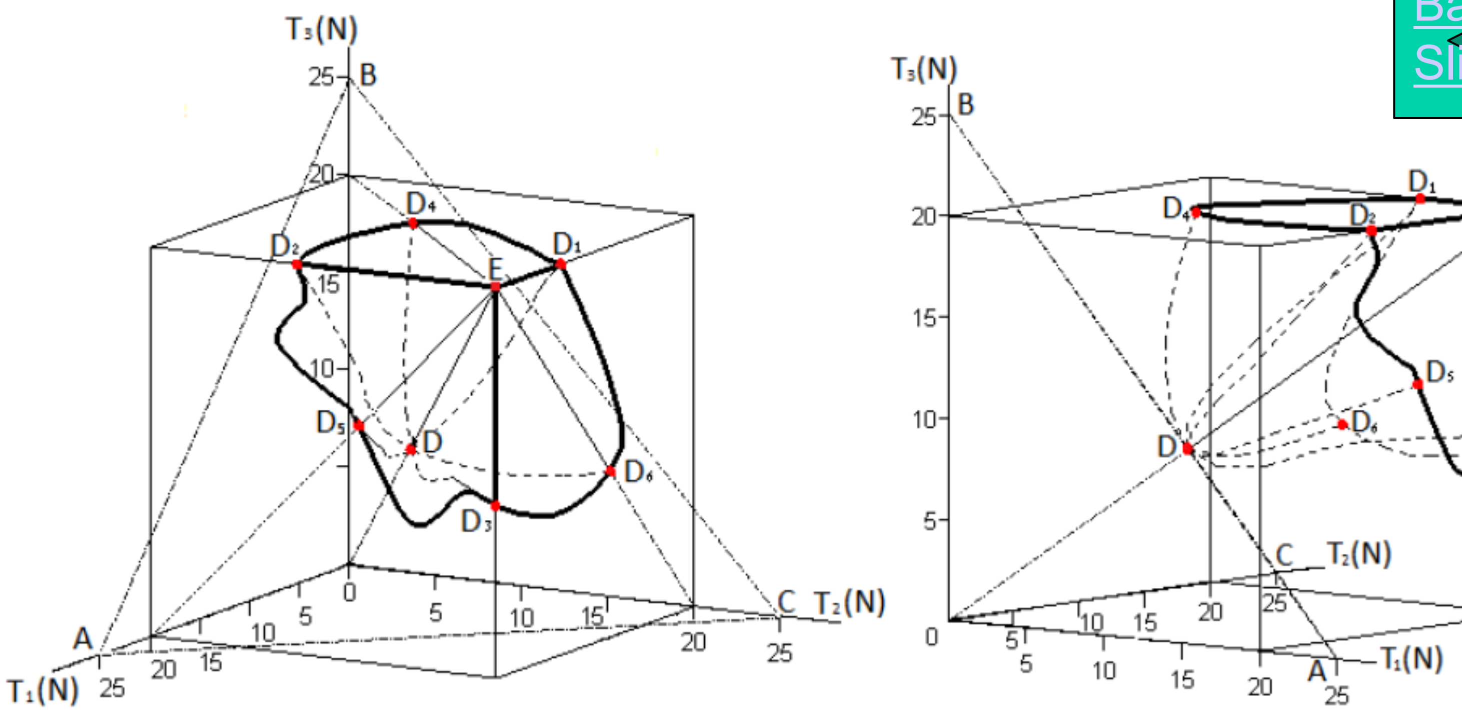
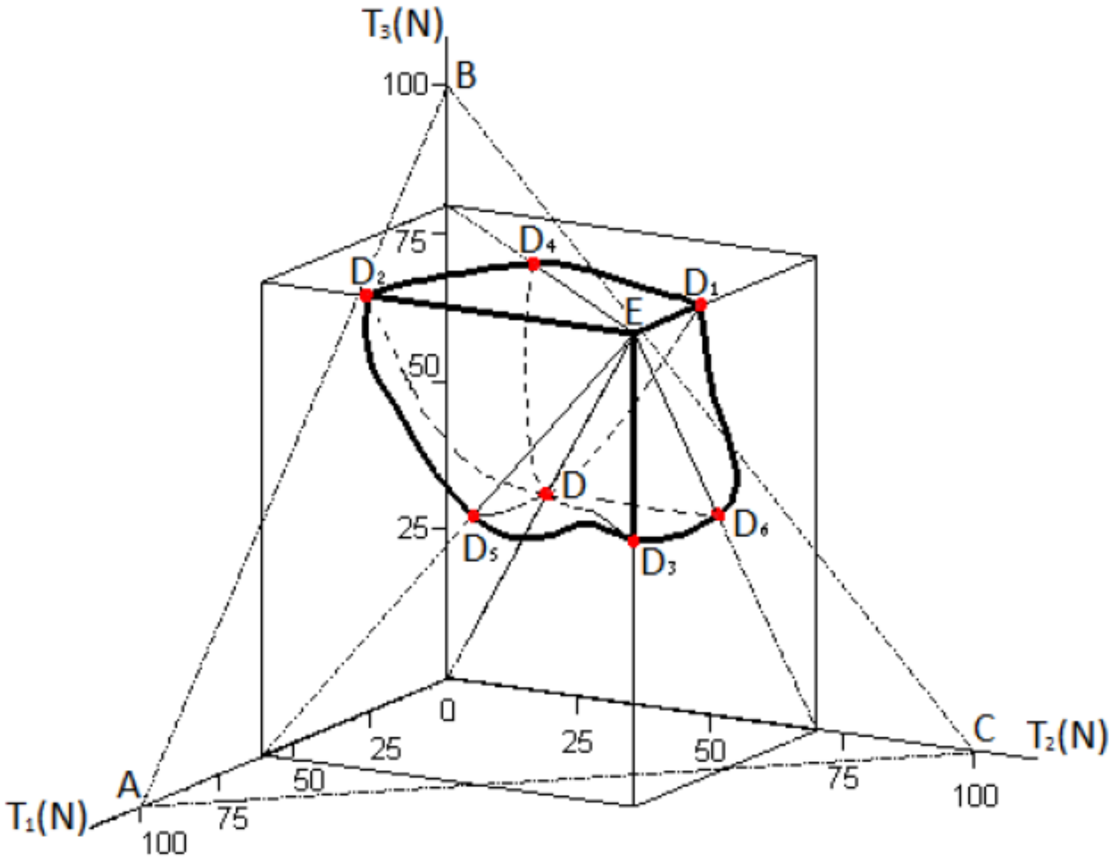


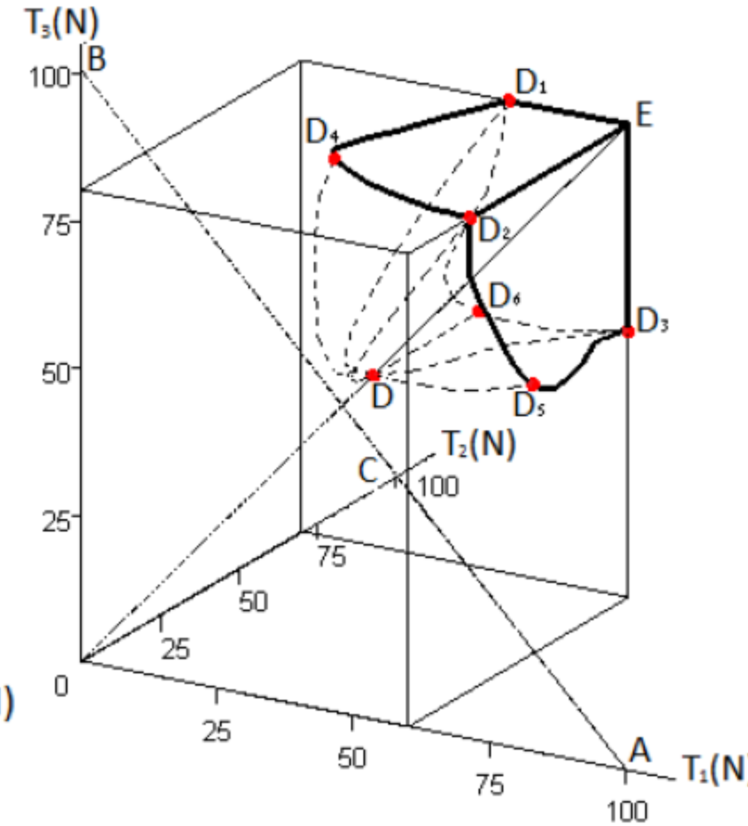
(a) View 1 (b) View 2

The tension workspace with an equilateral triangle payload and $\phi = 25°, \theta = 15°$ and $\psi = -5°$. The weight of the payload is $mg = 25N$. The payload capacities of three robots are $T_{imax} = 20N$ $(i = 1, 2, 3)$.

IK: Tension workspace



(a) View 1



(b) View 2

The tension workspace with a general payload and $\phi = 25°, \theta = 15°$ and $\psi = -5°$. The weight of the payload is $mg = 100N$. The payload capacities of three robots are respectively $T_{1max} = 60N$, $T_{2max} = 70N$ and $T_{3max} = 80N$.

Conclusions

(1) Direct Kinematics

- Analytic algorithm based on resultant elimination for planar 4-bar linkage
- Case studies with 3 to 6 cables

(2) Inverse Kinematics

- Analytic algorithm based on dialytic elimination (Up to 6 solutions for given tensions)
- Case studies for different payloads, tensions, orientations
- Tension workspace

(3) Stability Analysis

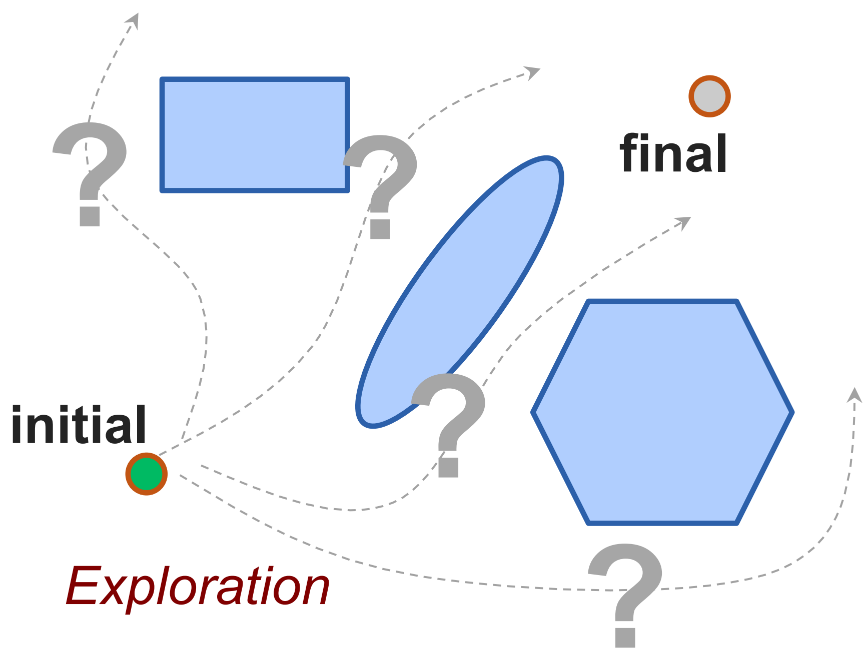
Jiang, Q., and Kumar, V., 2012, "Determination and Stability Analysis of Equilibrium Configurations of payloads Suspended from Multiple Aerial Robots",ASME Journal of Mechanisms and Robotics, Vol.4, No.2.

Jiang, Q., and Kumar, V., 2010, "The Inverse Kinematics of 3-D Towing", Proceedings of the 12th International Symposium: Advances in Robot Kinematics, June 27 – July 1, Piran-Portoroz, Slovenia.

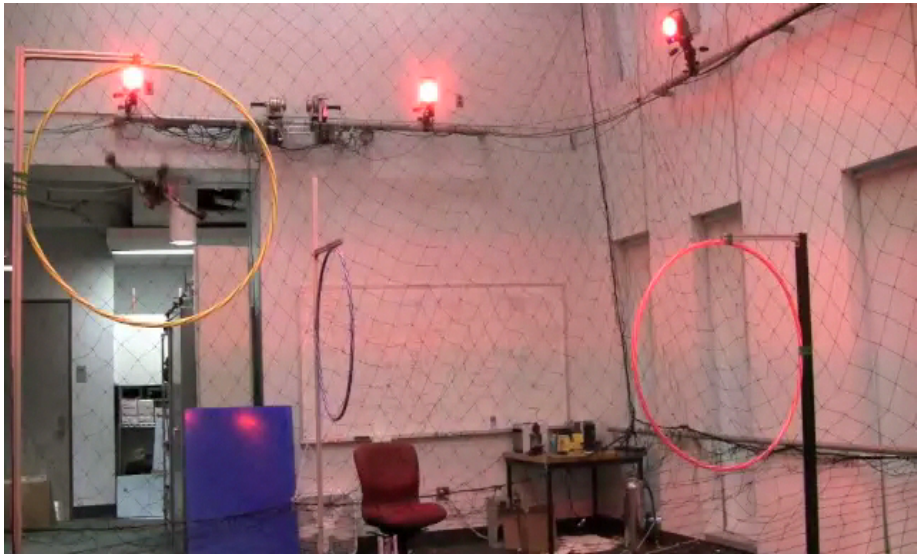
Jiang, Q., and Kumar, V., "The Inverse Kinematics of Cooperative Transport with Multiple Aerial Robots", accepted by IEEE Transactions on Robotics.

Homotopy Classes of Trajectories

- Coordinated motion planning for towing/skimming
- Finding geodesics (plans, controls) in complex spaces
- Exploration

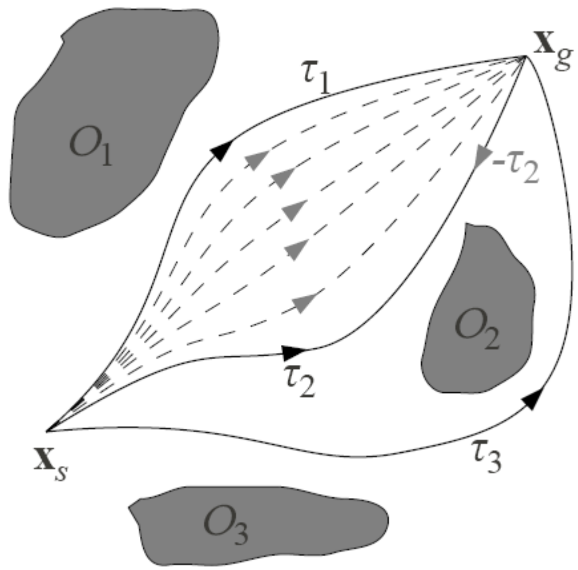


Exploration



Planning, optimal control

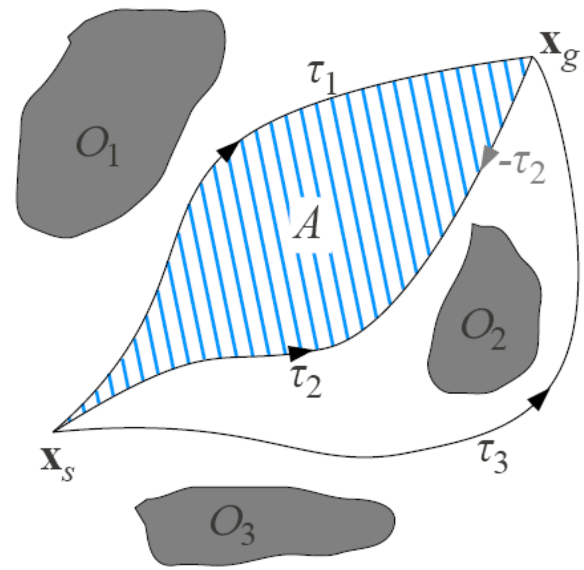
Homotopy and Homology



$\Longrightarrow$

$\Longleftarrow$

?



Homotopy

$\tau_1 \sim \tau_2$ τ_1 can be continuously deformed into τ_2

$\tau_2 \not\sim \tau_3$

Homology

$\tau_1 \sim \tau_2$ $\tau_1 \cup -\tau_2 = \partial A$

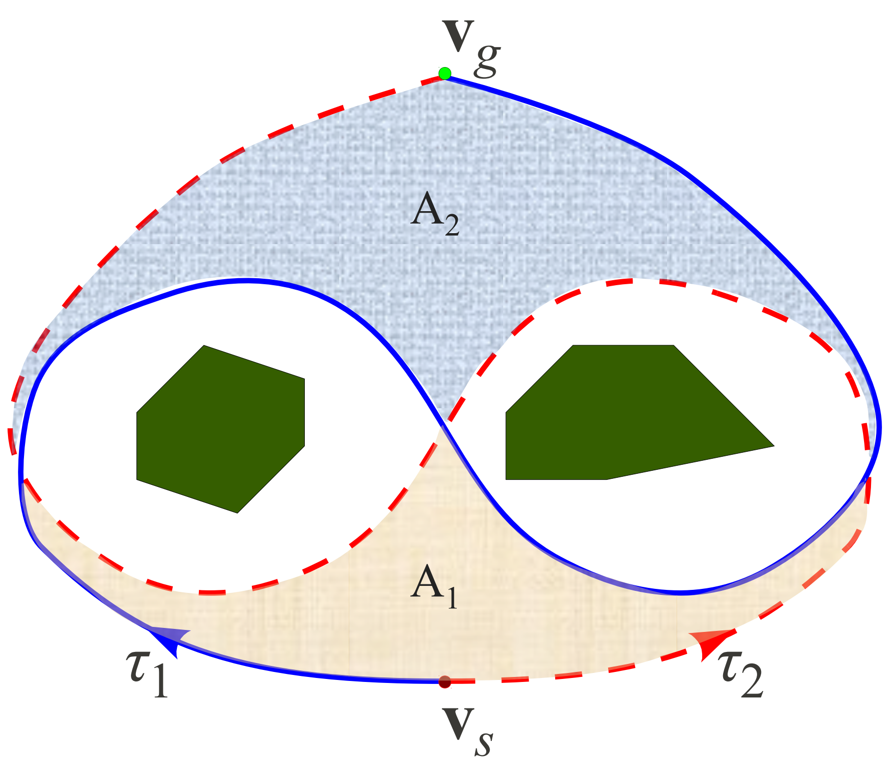
$\tau_2 \not\sim \tau_3$

Homotopy is easy to understand, but difficult to compute.

Homology groups can be computed (Hatcher, 2002)!

Penn Engineering

Homologous but not homotopic



$$\tau_1 \cup -\tau_2 = \partial A_1 \cup \partial A_2$$

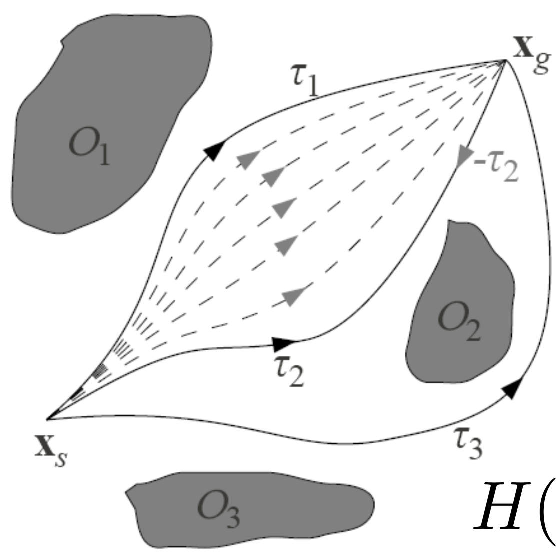
Homotopic implies homologous, but converse not necessarily true!

H-Signature

Find a 1-form whose integral along a trajectory encodes information about the homology (homotopy) class

$$H(\tau) = \int_\tau \omega$$

A homology (homotopy) class invariant for τ



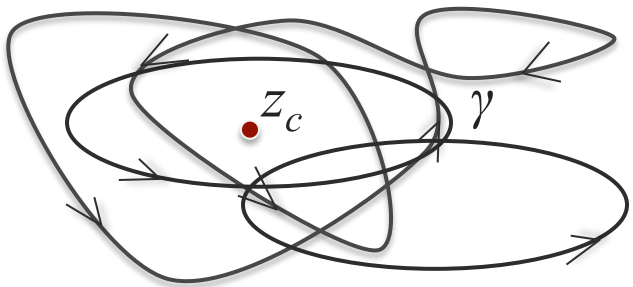
$$H(\tau_1) = H(\tau_2) \neq H(\tau_3)$$

$$H(\tau) = \int_\tau \omega$$

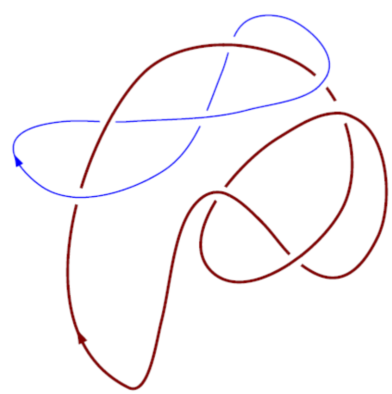
For single path-connected obstacle in two dimensions, the H-signature (homology class invariant) can by computed from the Cauchy Residue Theorem

Example: point obstacles in two-dimensional space

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_c}\, \mathrm{d}z = 0 \quad 2$$



Example: One dimensional obstacles in three-dimensional space (linking number)



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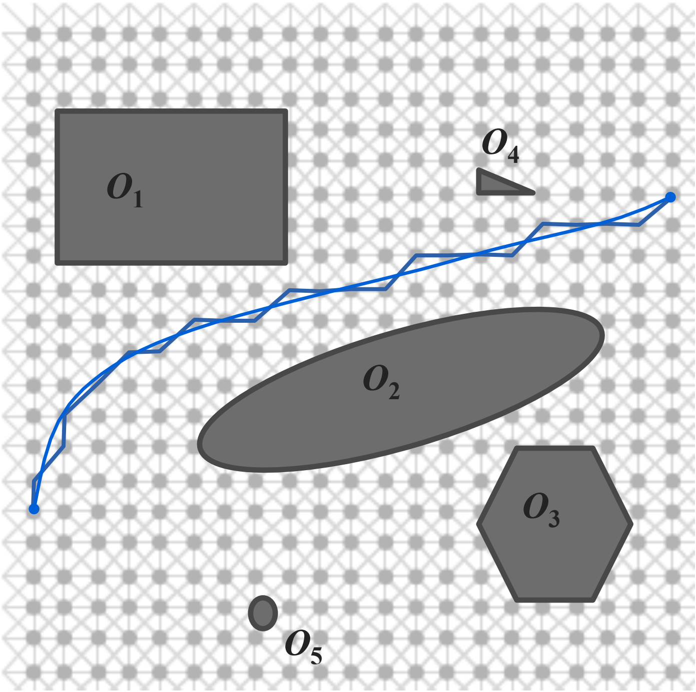
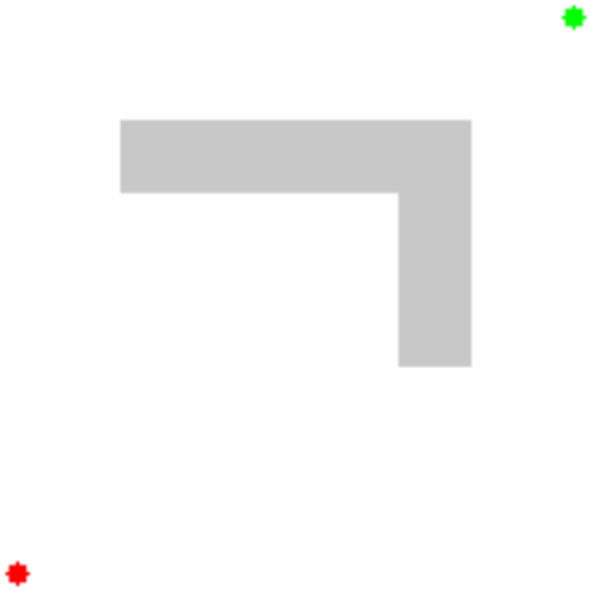
Homology classes and planning

Two Key ideas

1 H-signature to identify the **homology class of τ**.

2 Graph search to find trajectories





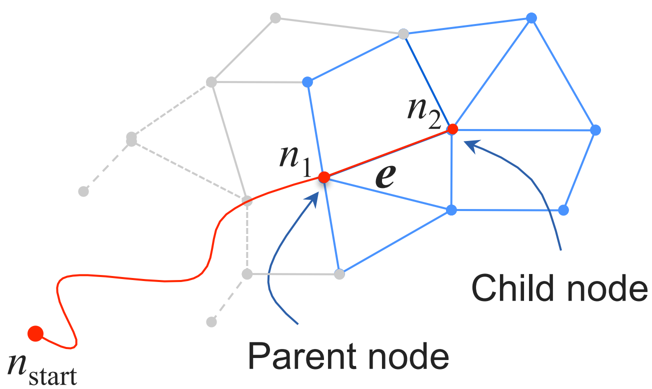
wikipedia.org

Homology classes and planning

Two Key ideas

1 H-signature to identify the **homology class of τ.**

2 Graph search to find trajectories



n_1 in "closed" list (expanded)

– next node to expand is n_2 ....

Cost

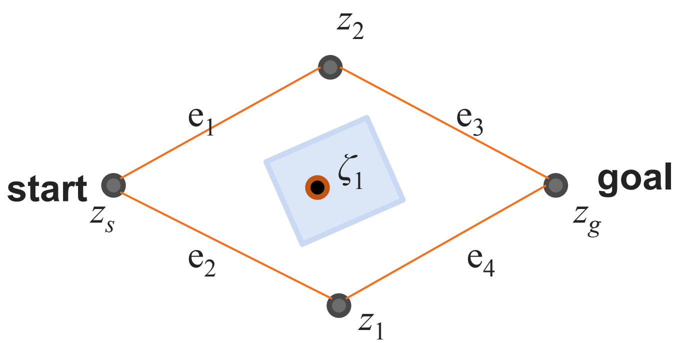
$$g(n_{\text{start}} n_2) = g(n_{\text{start}} n_1) + \text{cost}(\mathbf{e})$$

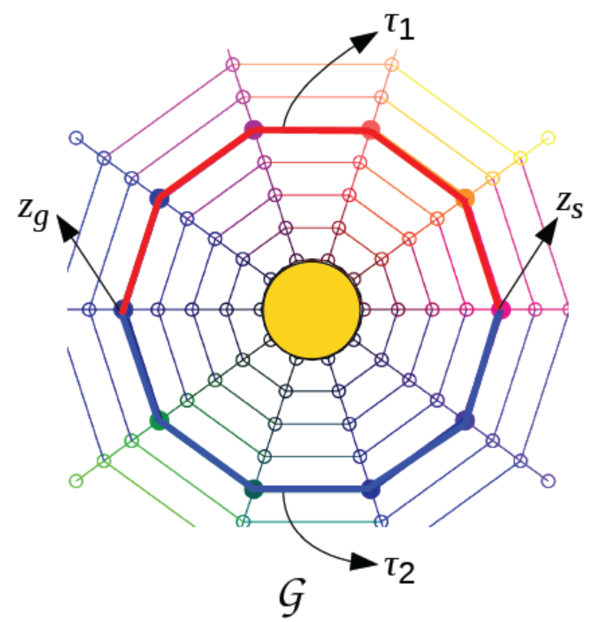
H-signature

$$H(n_{\text{start}} n_2) = H(n_{\text{start}} n_1) + H(\mathbf{e})$$

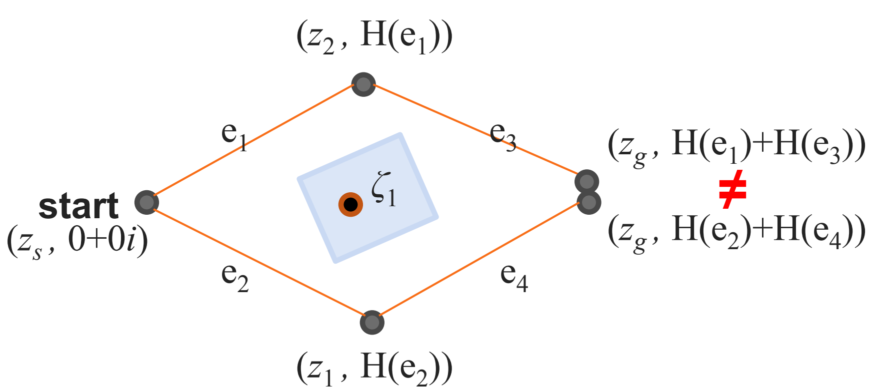
Find optimal paths with ***constraints on H***

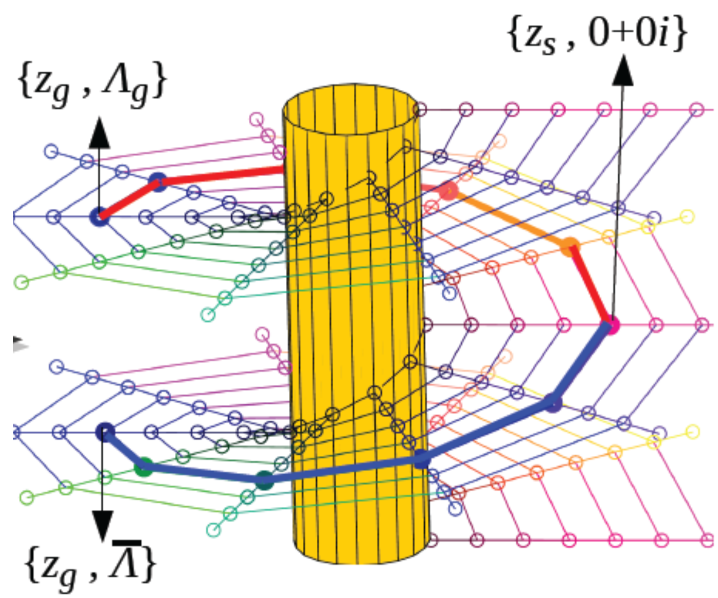
$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$





$$\mathcal{G}_H = (\mathcal{V}_H, \mathcal{E}_H)$$

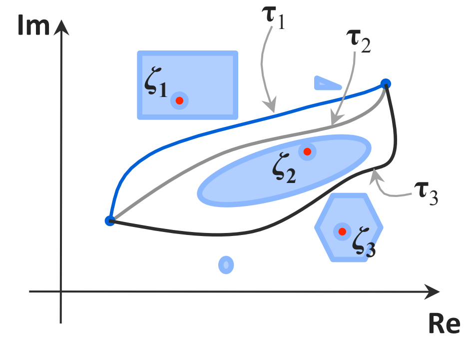




Planning in two dimensions

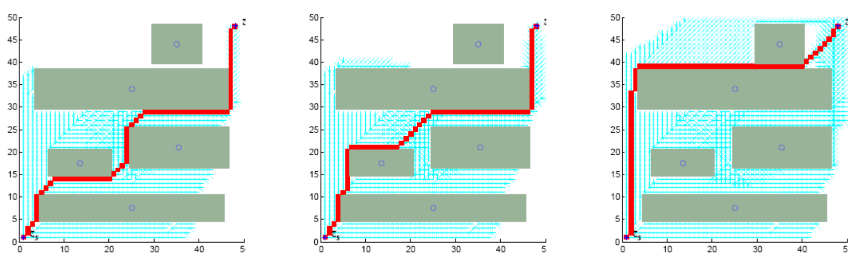
Construct a (vector) analytic function with singularities at “representative points” in the complex plane.

$$\mathcal{F}(z) = \begin{bmatrix} \frac{f_0(z)}{z-\zeta_1} \\ \frac{f_0(z)}{z-\zeta_1} \\ \vdots \\ \frac{f_0(z)}{z-\zeta_1} \end{bmatrix}$$

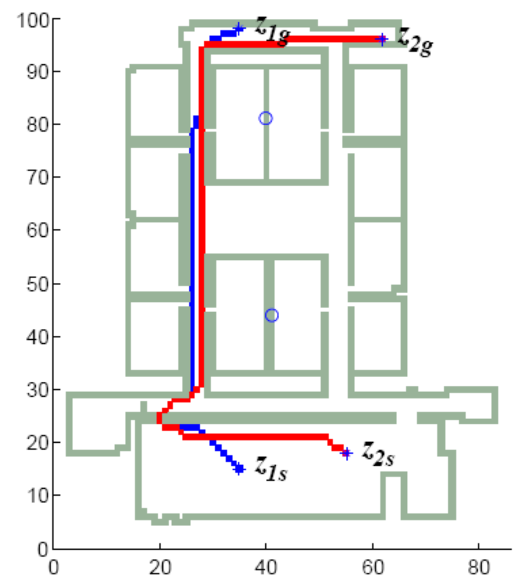


Leverage Cauchy Integral and Residue Theorems to design an additive *homotopy class invariant*.

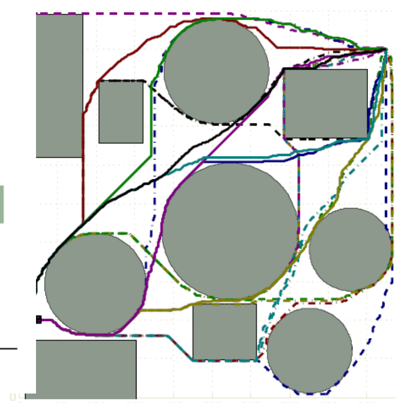
$$H(\tau) = \int_\tau \underbrace{\mathcal{F}(z)dz}_{\omega \text{ (diff. 1-form)}}$$



Graph-search based planning with homotopy class constraints.

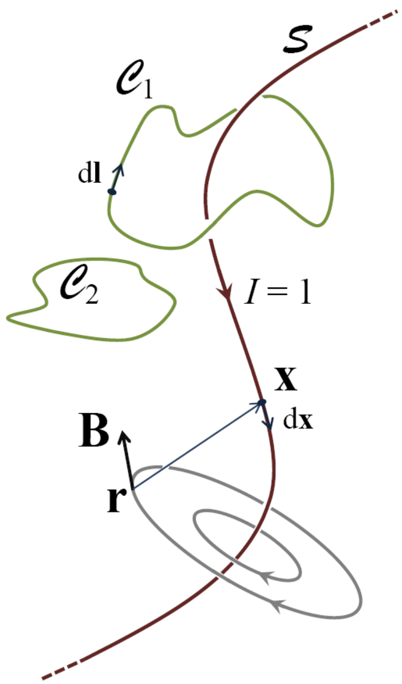


Optimal planning with homotopy class constraints *(visibility constraint)*



Homotopy class exploration in a large environment (1000x1000 discretized)

Planning in three dimensions

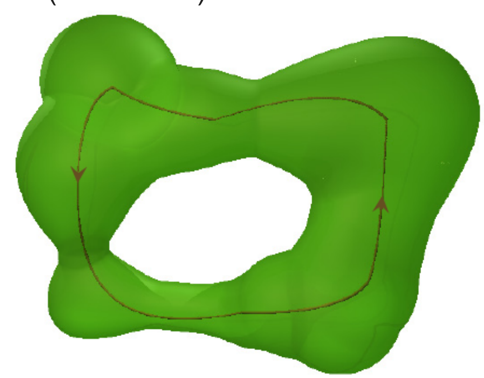


Biot-Savart's Law: $$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{S}} \frac{(\mathbf{x} - \mathbf{r}) \times d\mathbf{x}}{\|\mathbf{x} - \mathbf{r}\|^3}$$

Ampere's Law: $$\Xi(\mathcal{C}) := \int_{\mathcal{C}} \underbrace{\mathbf{B}(\mathbf{l}) \cdot d\mathbf{l}}_{\omega \text{ (diff. 1-form)}} = \mu_0 I_{encl}$$

$\mathbf{B}$: Magnetic field vector

μ_0: Magnetic constant (can be chosen as 1)



Skeletons of ***Simple Homotopy Inducing Obstacles*** are modeled as a ***current carrying conductors***

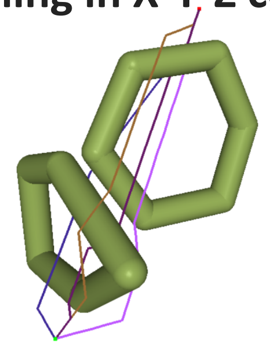
***h*-signature of trajectory *τ*:** $$\mathcal{H}(\tau) = [h_1(\tau),\ h_2(\tau),\ \ldots,\ h_M(\tau)]^T$$

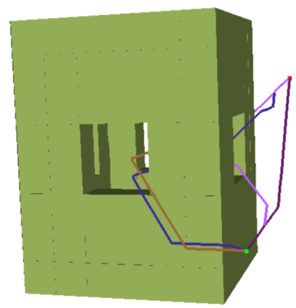
where, $$h_i(\tau) = \int_{\tau} \mathbf{B}_i(\mathbf{l}) \cdot d\mathbf{l} \quad , \quad \mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi} \int_{S_i} \frac{(\mathbf{x} - \mathbf{r}) \times d\mathbf{x}}{\|\mathbf{x} - \mathbf{r}\|^3}$$

[Bhattacharya *et al*, RSS 2011]

Results in 3-D

Planning in X-Y-Z configuration space:

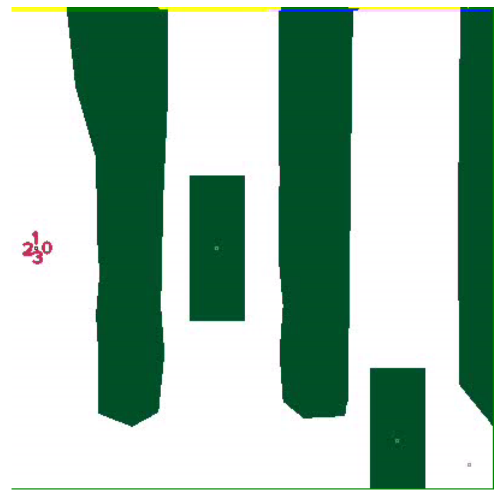


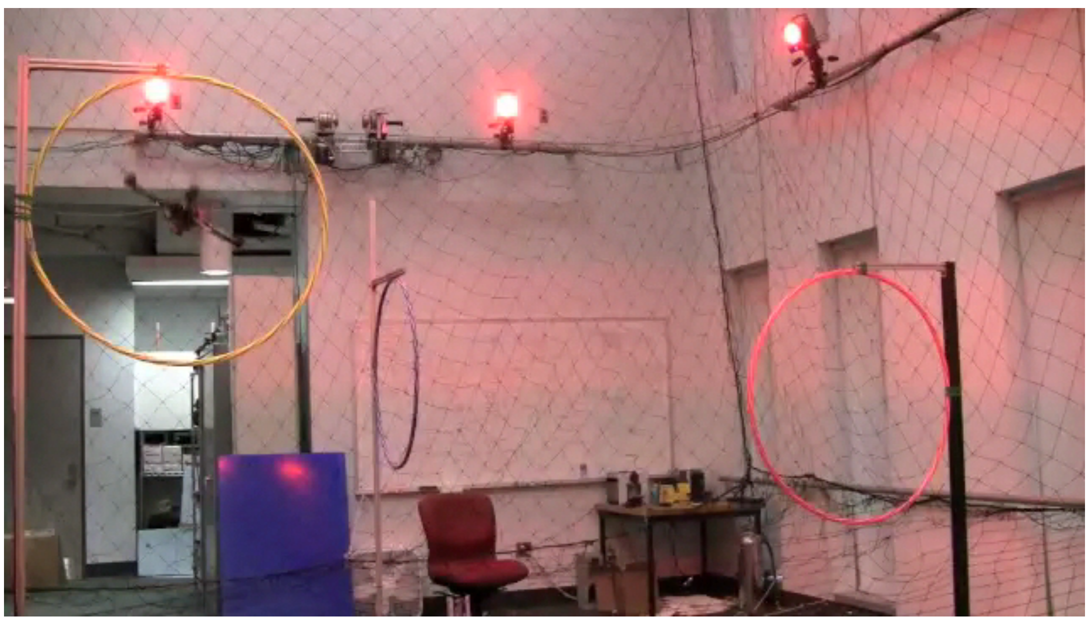


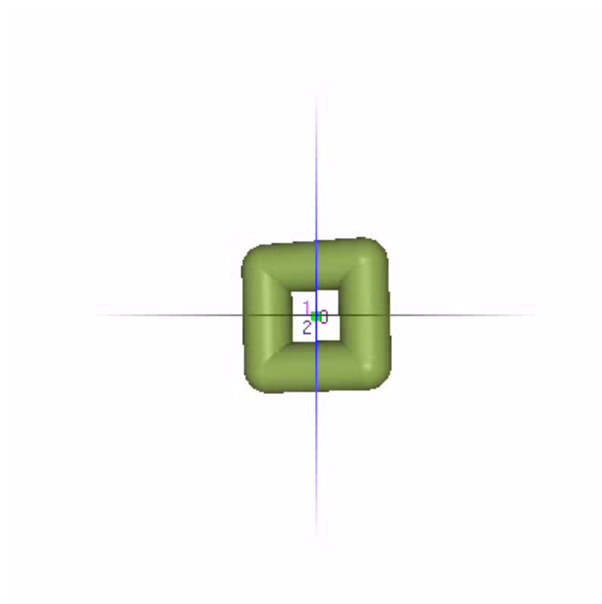
Exploration of 4 homotopy classes in presence of 2 SHIOs

Exploration of 4 homotopy classes in presence of 4 SHIOs

Planning in Space Time







Optimal control with space time constraints (Mellinger and Kumar, 2011)

Linking Number in D-Dimensional Euclidean Spaces

Key idea

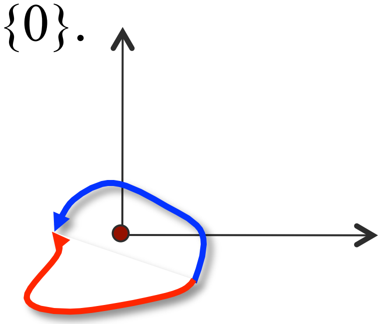
Construct *S*, a (*D*-2)-dimensional ***homotopy equivalent*** of an obstacle

Find a ***differential 1-form*** that, when integrated along a closed curve, gives its ***linking number*** with *S*.

- Establish a surjective map between $\mathbf{R}^D$-S and $\mathbf{R}^D$-$\{0\}$ and exploit the known formulae for closed but non-exact differential forms in $\mathbf{R}^D$-$\{0\}$.

Example (D=2)

$$d\theta = \frac{1}{x^2+y^2}(-ydx + xdy)$$



$$\eta(\mathbf{s}) = \sum_{k=1}^{D} \mathcal{G}_k(\mathbf{s}) \;\; (-1)^{k+1} \;\; \mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{k-1} \wedge \mathrm{d}s_{k+1} \wedge \cdots \wedge \mathrm{d}s_D$$

$$\mathcal{G}_k(\mathbf{s}) = \frac{1}{A_{D-1}} \frac{s_k}{(s_1^2 + s_2^2 + \cdots + s_D^2)^{D/2}}$$

Linking Number in Punctured Euclidean Spaces

Multiple Obstacles

Find a ***differential 1-form*** that, when integrated along a closed curve, gives its ***linking number*** with *S*.

- Establish a surjective map between $\mathbf{R}^D$ -S and $\mathbf{R}^D$ -$\{0\}$ and exploit the known formulae for closed but non-exact differential forms in $\mathbf{R}^D$ -$\{0\}$.
- Decompose *S* ((*D*-2)-dimensional skeleton) into M connected components:

$$S_1 \sqcup S_2 \sqcup \cdots \sqcup S_M = S$$

$$\omega_i = \sum_{k=1}^{D} \sum_{\substack{j=1 \\ j \neq k}}^{D} U_j^k(\mathbf{x}; S_i)\ \mathrm{d}x_j \qquad \mathcal{H}(\tau) = \int_\tau \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_M \end{bmatrix}$$

$$U_j^k(\mathbf{x}; S) = (-1)^{k-j-1-\mathbf{is}(j<k)} \int_S \mathcal{G}_k(\mathbf{x} - \mathbf{x}')\ \mathrm{d}x'_1 \wedge \mathrm{d}x'_2 \cdots \wedge \widehat{x'_j, x'_k} \wedge \cdots \wedge \mathrm{d}x'_D$$

$$\mathcal{G}_k(\mathbf{s}) = \frac{1}{A_{D-1}} \frac{s_k}{\left(s_1^2 + s_2^2 + \cdots + s_D^2\right)^{D/2}}$$

[*Bhattacharya, Lipsky, Ghrist and Kumar*, 2012 (submitted)]

Problem 1

Generate ***optimal*** trajectory with ***homology class constraints***

Optimization

Minimize cost functional

$$\min_{q(t)} \int_0^1 \mathcal{L}\left(q,\ \dot{q}, \ldots, q^{(r)}\right) dt$$

Non convex

Constraints

Easy to compute

Trajectory belonging to a specified homology class

$$H(q(1)) = H_{\mathrm{des}}$$

Problem 2

Generate ***optimal*** trajectory with ***homotopy class constraints***

Optimization

Minimize cost functional

$$\min_{q(t)} \int_0^1 \mathcal{L}\left(q,\ \dot{q}, \ldots, q^{(r)}\right) dt$$

Harder to compute

Constraints

Trajectory belonging to a specified homotopy class

Assumptions

1. Polygonal Obstacles
2. Quadratic Cost

[Kim, Bhattacharya, Sreenath, Kumar, ARK 2012]

Conclusion

Geometry, kinematics and statics of cable-driven systems introduce challenges and opportunities

- Homotopy classes (and homology classes)
- Instantaneous kinematics
- Direct and Inverse kinematics
- Dynamics and control
- Scaling up to large numbers