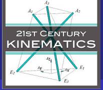
Kinematics and Algebraic Geometry

Manfred L. Husty   Hans-Peter Schröcker

Institute of Basic Sciences in Engineering, Unit Geometry and CAD,
University Innsbruck, Austria
manfred.husty@uibk.ac.at

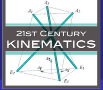
Workshop on 21st Century Kinematics, Chicago 2012



Outline of Lecture

Introduction

Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).



Introduction

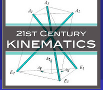
Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).

- Within CK one tries to answer fundamental questions arising in the ***analysis and synthesis of kinematic chains.***

Introduction

Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).

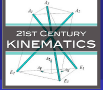
- Within CK one tries to answer fundamental questions arising in the ***analysis and synthesis of kinematic chains.***
- Kinematic chains are constituent elements of ***serial or parallel robots, wired robots, humanoid robots, walking and jumping machines or rolling and autonomous robots.***



Introduction

Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).

- Within CK one tries to answer fundamental questions arising in the ***analysis and synthesis of kinematic chains.***
- Kinematic chains are constituent elements of ***serial or parallel robots, wired robots, humanoid robots, walking and jumping machines or rolling and autonomous robots.***
- The fundamental questions, going far beyond the classical kinematics involve the number of solutions, complex or real to, for example, ***forward or inverse kinematics***, the description of ***singular solutions***, the mathematical solution of ***workspace or synthesis*** questions.



Introduction

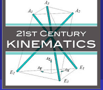
Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).

- Within CK one tries to answer fundamental questions arising in the ***analysis and synthesis of kinematic chains.***
- Kinematic chains are constituent elements of ***serial or parallel robots, wired robots, humanoid robots, walking and jumping machines or rolling and autonomous robots.***
- The fundamental questions, going far beyond the classical kinematics involve the number of solutions, complex or real to, for example, ***forward or inverse kinematics***, the description of ***singular solutions***, the mathematical solution of ***workspace or synthesis*** questions.
- Such problems are often described by ***systems of multivariate algebraic or functional equations*** and it turns out that even relatively simple kinematic problems involving multi-parameter systems lead to complicated nonlinear equations.

Introduction

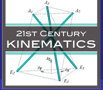
Computational Kinematics is that branch of kinematics which involves intensive computations not only of numerical type but also of symbolic nature (Angeles 1993).

- Within CK one tries to answer fundamental questions arising in the ***analysis and synthesis of kinematic chains.***
- Kinematic chains are constituent elements of ***serial or parallel robots, wired robots, humanoid robots, walking and jumping machines or rolling and autonomous robots.***
- The fundamental questions, going far beyond the classical kinematics involve the number of solutions, complex or real to, for example, ***forward or inverse kinematics***, the description of ***singular solutions***, the mathematical solution of ***workspace or synthesis*** questions.
- Such problems are often described by ***systems of multivariate algebraic or functional equations*** and it turns out that even relatively simple kinematic problems involving multi-parameter systems lead to complicated nonlinear equations.
- Geometric insight and geometric preprocessing are often key to the solution



Introduction

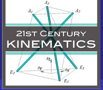
Analytic description of kinematic chains:



Introduction

Analytic description of kinematic chains:

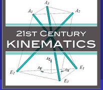
- Parametric and implicit representations



Introduction

Analytic description of kinematic chains:

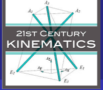
- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)



Introduction

Analytic description of kinematic chains:

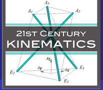
- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains



Introduction

Analytic description of kinematic chains:

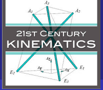
- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained



Introduction

Analytic description of kinematic chains:

- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained
- Complete analysis and synthesis needs all solutions



Introduction

Analytic description of kinematic chains:

- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained
- Complete analysis and synthesis needs all solutions
- We propose the use of algebraic constraint equations, as to be able to use strong methods and algorithms from algebraic geometry

Introduction

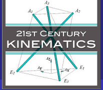
Analytic description of kinematic chains:

- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained
- Complete analysis and synthesis needs all solutions
- We propose the use of algebraic constraint equations, as to be able to use strong methods and algorithms from algebraic geometry
- An important task is to find the simplest algebraic constraint equations, that describe the chains.

Introduction

Analytic description of kinematic chains:

- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained
- Complete analysis and synthesis needs all solutions
- We propose the use of algebraic constraint equations, as to be able to use strong methods and algorithms from algebraic geometry
- An important task is to find the simplest algebraic constraint equations, that describe the chains.
- Geometric and algebraic preprocessing is needed before elimination, Gröbner base computation or numerical solution process starts



Introduction

Analytic description of kinematic chains:

- Parametric and implicit representations
- Different parametrizations of the displacement group $SE(3)$ (Euler angles, Rodrigues parameters, Euler parameters, Study parameters, quaternions, dual quaternions)
- Most the time vector loop equations are used to describe the chains
- Very often only a single numerical solution is obtained
- Complete analysis and synthesis needs all solutions
- We propose the use of algebraic constraint equations, as to be able to use strong methods and algorithms from algebraic geometry
- An important task is to find the simplest algebraic constraint equations, that describe the chains.
- Geometric and algebraic preprocessing is needed before elimination, Gröbner base computation or numerical solution process starts
- Algebraic constraint equations yield answers to the overall behavior of a kinematic chain $\rightarrow$ **Global Kinematics**

Introduction


H-P. Schröcker


D. Walter


M. Pfurner


F. Pernkopf


K. Brunnthaler


J. Schadlbauer

Mehdi Tale Masouleh, Clément Gosselin (Laval University, Quebec City)

J.M. Selig (London, UK)

P. Zsombor-Murray, M. J. D. Hayes (McGill, Montreal)

A. Karger (Charles University Prag, Czech Republic)

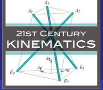
Introduction

In the following I want to show

Introduction

In the following I want to show

- Some algebraic basics of kinematics



Introduction

In the following I want to show

- Some algebraic basics of kinematics
- How algebraic constraint equations can be obtained from parametric equations involving sines and cosines

Introduction

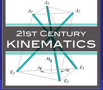
In the following I want to show

- Some algebraic basics of kinematics
- How algebraic constraint equations can be obtained from parametric equations involving sines and cosines
- How freedom of mechanisms can be formulated within this frame

Introduction

In the following I want to show

- Some algebraic basics of kinematics
- How algebraic constraint equations can be obtained from parametric equations involving sines and cosines
- How freedom of mechanisms can be formulated within this frame
- How the same equations can be used for analysis and synthesis



Introduction

In the following I want to show

- Some algebraic basics of kinematics
- How algebraic constraint equations can be obtained from parametric equations involving sines and cosines
- How freedom of mechanisms can be formulated within this frame
- How the same equations can be used for analysis and synthesis
- How singularities can be obtained within the algebraic formulation

Introduction

In the following I want to show

- Some algebraic basics of kinematics
- How algebraic constraint equations can be obtained from parametric equations involving sines and cosines
- How freedom of mechanisms can be formulated within this frame
- How the same equations can be used for analysis and synthesis
- How singularities can be obtained within the algebraic formulation
- How this framework can be used for the analysis of lower dof parallel manipulators

Kinematic mapping

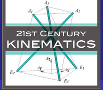
Euclidean displacement:

$$\gamma\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{a} \tag{1}$$

$\mathbf{A}$ proper orthogonal 3×3 matrix, $\mathbf{a} \in \mathbb{R}^3 \ldots$ vector

group of Euclidean displacements: $\mathrm{SE}(3)$

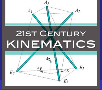
$$\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & \mathbf{o}^T \\ \mathbf{a} & \mathbf{A} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}. \tag{2}$$



Kinematic mapping

Study's kinematic mapping $\varkappa$:

$$\varkappa : \alpha \in \mathrm{SE}(3) \mapsto \mathbf{x} \in \mathbb{P}^7$$



Kinematic mapping

Study's kinematic mapping $\varkappa$:

$$\varkappa : \alpha \in \mathrm{SE}(3) \mapsto \mathbf{x} \in \mathbb{P}^7$$

pre-image of $\mathbf{x}$ is the displacement α

$$\frac{1}{\Delta}\begin{bmatrix} \Delta & 0 & 0 & 0 \\ p & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ q & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ r & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{3}$$

$$\begin{aligned} p &= 2(-x_0y_1+x_1y_0-x_2y_3+x_3y_2), \\ q &= 2(-x_0y_2+x_1y_3+x_2y_0-x_3y_1), \\ r &= 2(-x_0y_3-x_1y_2+x_2y_1+x_3y_0), \end{aligned} \tag{4}$$

$\Delta = x_0^2+x_1^2+x_2^2+x_3^2.$

Kinematic mapping

Study's kinematic mapping $\varkappa$:

$$\varkappa : \alpha \in \mathrm{SE}(3) \mapsto \mathbf{x} \in \mathbb{P}^7$$

pre-image of $\mathbf{x}$ is the displacement α

$$\frac{1}{\Delta}\begin{bmatrix} \Delta & 0 & 0 & 0 \\ p & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ q & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ r & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{3}$$

$$\begin{aligned} p &= 2(-x_0y_1+x_1y_0-x_2y_3+x_3y_2), \\ q &= 2(-x_0y_2+x_1y_3+x_2y_0-x_3y_1), \\ r &= 2(-x_0y_3-x_1y_2+x_2y_1+x_3y_0), \end{aligned} \tag{4}$$

$\Delta = x_0^2+x_1^2+x_2^2+x_3^2$.

$$S_6^2: \quad x_0y_0+x_1y_1+x_2y_2+x_3y_3 = 0, \qquad x_i \quad \text{not all} \quad 0 \tag{5}$$

$[x_0 : \cdots : y_3]^T$ *Study parameters* = parametrization of $SE(3)$ with dual quaternions

Kinematic mapping

How do we get the Study parameters when a proper orthogonal matrix $\mathbf{A} = [a_{ij}]$ and the translation vector $\mathbf{a} = [a_k]^T$ are given?

Kinematic mapping

How do we get the Study parameters when a proper orthogonal matrix $\mathbf{A} = [a_{ij}]$ and the translation vector $\mathbf{a} = [a_k]^T$ are given?

Cayley map, not singularity free (180°)

Kinematic mapping

How do we get the Study parameters when a proper orthogonal matrix $\mathbf{A} = [a_{ij}]$ and the translation vector $\mathbf{a} = [a_k]^T$ are given?

Cayley map, not singularity free (180°)

Rotation part:

$$
\begin{aligned}
x_0 : x_1 : x_2 : x_3 &= 1 + a_{11} + a_{22} + a_{33} : a_{32} - a_{23} : a_{13} - a_{31} : a_{21} - a_{12} \\
&= a_{32} - a_{23} : 1 + a_{11} - a_{22} - a_{33} : a_{12} + a_{21} : a_{31} + a_{13} \\
&= a_{13} - a_{31} : a_{12} + a_{21} : 1 - a_{11} + a_{22} - a_{33} : a_{23} + a_{32} \\
&= a_{21} - a_{12} : a_{31} + a_{13} : a_{23} - a_{32} : 1 - a_{11} - a_{22} + a_{33}
\end{aligned}
\tag{6}
$$

Translation part:

$$
\begin{aligned}
2y_0 &= a_1x_1 + a_2x_2 + a_3x_3, & 2y_1 &= -a_1x_0 + a_3x_2 - a_2x_3, \\
2y_2 &= -a_2x_0 - a_3x_1 + a_1x_3, & 2y_3 &= -a_3x_0 + a_2x_1 - a_1x_2.
\end{aligned}
\tag{7}
$$

Remark: some people have been working on this topic like

E. Study, W. Blaschke, E.A. Weiss, ...

A. Yang, B. Roth, B. Ravani (and his students), A. Karger, W. Ströher, H. Stachel,....

sometimes using different names like Clifford Algebra:
M. McCarthy...

Remark: some people have been working on this topic like

E. Study, W. Blaschke, E.A. Weiss, ...

A. Yang, B. Roth, B. Ravani (and his students), A. Karger, W. Ströher, H. Stachel,....

sometimes using different names like Clifford Algebra:
M. McCarthy...

Example:
A rotation about the z-axis through the angle φ is described by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Its kinematic image, computed via (6) and (7) is

$$\mathbf{r} = [1+\cos\varphi : 0 : 0 : \sin\varphi : 0 : 0 : 0 : 0]^T. \tag{9}$$

As φ varies in $[0, 2\pi)$, $\mathbf{r}$ describes a straight line on the Study quadric which reads after algebraization

$$\mathbf{r} = [1 : 0 : 0 : u : 0 : 0 : 0 : 0]^T. \tag{10}$$

A special one parameter motion is defined by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ \sin\frac{t}{2} & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Its kinematic image, computed via (6) and (7) reads

$$\mathbf{r} = \left[2+2\cos t : 0 : 0 : 2\sin t : \sin\frac{t}{2}\sin t : 0 : 0 : -\frac{1}{2}\sin\frac{t}{2}(2+2\cos t)\right] \tag{12}$$

After algebraization and some manipulation we obtain

$$\mathbf{r} = [-1+u^4 : 0 : 0 : -2u(1+u^2) : 2u^2 : 0 : 0 : u(1-u^2)], \tag{13}$$

represents a rational curve of degree four on the Study quadric.

The motion corresponding to this curve is a special case of the well known Bricard motions where all point-paths are spherical curves.

Kinematic mapping

S_6^2 is called Study quadric

Kinematic mapping

S_6^2 is called Study quadric

the map between S_6^2 and $SE(3)$ is not one to one,

$$F: x_0 = x_1 = x_2 = x_3 = 0, \quad E: y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0. \tag{14}$$

Exceptional generator F, exceptional quadric E

(these things come from the circle points in Euclidean geometry!)

Planar displacements: $x_2 = x_3 = 0, y_0 = y_1 = 0$

$$\frac{1}{x_0^2 + x_3^2}\begin{bmatrix} x_0^2 + x_3^2 & 0 & 0 \\ -2(x_0 y_1 - x_3 y_2) & x_0^2 - x_3^2 & -2x_0 x_3 \\ -2(x_0 y_2 + x_3 y_1) & 2x_0 x_3 & x_0^2 - x_3^2 \end{bmatrix}$$

$\mathrm{SE}(2)$ (we omit the last row and the last column)

Spherical displacements: $y_i = 0$

$$\frac{1}{\Delta}\begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1 x_2 - x_0 x_3) & 2(x_1 x_3 + x_0 x_2) \\ 2(x_1 x_2 + x_0 x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2 x_3 - x_0 x_1) \\ 2(x_1 x_3 - x_0 x_2) & 2(x_2 x_3 + x_0 x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix} \tag{15}$$

where $\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2$. $\rightarrow$ $SO^+(3)$

generate 3-spaces on S_6^2

Planar displacements: $x_2 = x_3 = 0, y_0 = y_1 = 0$

$$\frac{1}{x_0^2 + x_3^2}\begin{bmatrix} x_0^2 + x_3^2 & 0 & 0 \\ -2(x_0y_1 - x_3y_2) & x_0^2 - x_3^2 & -2x_0x_3 \\ -2(x_0y_2 + x_3y_1) & 2x_0x_3 & x_0^2 - x_3^2 \end{bmatrix}$$

$\mathrm{SE}(2)$ (we omit the last row and the last column)

Spherical displacements: $y_i = 0$

$$\frac{1}{\Delta}\begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix} \tag{15}$$

where $\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2$. $\to$ $SO^+(3)$

generate 3-spaces on S_6^2
more properties:

J. Selig, Geometric Fundamentals of Robotics, 2nd. ed. Springer 2005
H., Pfurner, Schröcker, Brunnthaler. Algebraic methods in mechanism analysis and synthesis. Robotica, 25(6):661-675, 2007.

Quaternions

The set of quaternions $\mathbb{H}$ is the vector space $\mathbb{R}^4$ together with the quaternion multiplication

$$
\begin{aligned}
(a_0, a_1, a_2, a_3) \star (b_0, b_1, b_2, b_3) = (&a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3,\\
&a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2,\\
&a_0 b_2 - a_1 b_3 + a_2 b_0 - a_3 b_1,\\
&a_0 b_3 - a_1 b_2 - a_2 b_1 + a_3 b_0).
\end{aligned}
\tag{16}
$$

The triple $(\mathbb{H}, +, \star)$ (with component wise addition) forms a skew field. The real numbers can be embedded into this field via $x \mapsto (x, 0, 0, 0)$, and vectors $\mathbf{x} \in \mathbb{R}^3$ are identified with quaternions of the shape $(0, \mathbf{x})$.

Every quaternion is a unique linear combination of the four basis quaternions $\mathbf{1} = (1, 0, 0, 0)$, $\mathbf{i} = (0, 1, 0, 0)$, $\mathbf{j} = (0, 0, 1, 0)$, and $\mathbf{k} = (0, 0, 0, 1)$.

The multiplication table is

$\star$	**1**	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	$-$**1**	**k**	$-$**j**
j	**j**	$-$**k**	$-$**1**	**i**
k	**k**	**j**	$-$**i**	$-$**1**

Conjugate quaternion and *norm* are defined as

$$\overline{A} = (a_0, -a_1, -a_2, -a_3), \quad \|A\| = \sqrt{A \star \overline{A}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}. \qquad (17)$$

Quaternions are closely related to spherical kinematic mapping.

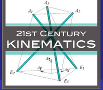
Consider a vector $\mathbf{a} = [a_1, a_2, a_3]^T$ and a matrix $\mathbf{X}$ of the shape (15).

The product $\mathbf{b} = \mathbf{X} \cdot \mathbf{a}$ can also be written as

$$B = X \star A \star \overline{X} \tag{18}$$

where $X = (x_0, x_1, x_2, x_3)$, $\|X\| = 1$ and $A = (0, \mathbf{a})$, $B = (0, \mathbf{b})$.

From this follows:



Quaternions are closely related to spherical kinematic mapping.

Consider a vector $\mathbf{a} = [a_1, a_2, a_3]^T$ and a matrix $\mathbf{X}$ of the shape (15).

The product $\mathbf{b} = \mathbf{X} \cdot \mathbf{a}$ can also be written as

$$B = X \star A \star \overline{X} \tag{18}$$

where $X = (x_0, x_1, x_2, x_3)$, $\|X\| = 1$ and $A = (0, \mathbf{a})$, $B = (0, \mathbf{b})$.

From this follows:

Spherical displacements can also be described by *unit quaternions* and *spherical kinematic mapping* maps a spherical displacement to the corresponding *unit quaternion*.

To describe general Euclidean displacements extend the concept of quaternions.

A *dual quaternion* Q is a quaternion over the ring of dual numbers

$$Q = Q_0 + \varepsilon Q_1, \tag{19}$$

where $\varepsilon^2 = 0$.

The algebra of dual quaternions has eight basis elements **1**, **i**, **j**, **k**, ε, ε**i**, ε**j**, and ε**k** and the multiplication table

$\star$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	ε	$\varepsilon\mathbf{i}$	$\varepsilon\mathbf{j}$	$\varepsilon\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	ε	$\varepsilon\mathbf{i}$	$\varepsilon\mathbf{j}$	$\varepsilon\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$\varepsilon\mathbf{i}$	$-\varepsilon\mathbf{1}$	$\varepsilon\mathbf{k}$	$-\varepsilon\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$	$\varepsilon\mathbf{j}$	$-\varepsilon\mathbf{k}$	$-\varepsilon\mathbf{1}$	$\varepsilon\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$	$\varepsilon\mathbf{k}$	$\varepsilon\mathbf{j}$	$-\varepsilon\mathbf{i}$	$-\varepsilon\mathbf{1}$
$\varepsilon\mathbf{1}$	ε	$\varepsilon\mathbf{i}$	$\varepsilon\mathbf{j}$	$\varepsilon\mathbf{k}$	0	0	0	0
$\varepsilon\mathbf{i}$	$\varepsilon\mathbf{i}$	$-\varepsilon\mathbf{1}$	$\varepsilon\mathbf{k}$	$-\varepsilon\mathbf{j}$	0	0	0	0
$\varepsilon\mathbf{j}$	$\varepsilon\mathbf{j}$	$-\varepsilon\mathbf{k}$	$-\varepsilon\mathbf{1}$	$\varepsilon\mathbf{i}$	0	0	0	0
$\varepsilon\mathbf{k}$	$\varepsilon\mathbf{k}$	$\varepsilon\mathbf{j}$	$-\varepsilon\mathbf{i}$	$-\varepsilon\mathbf{1}$	0	0	0	0

Dual quaternions know two types of conjugation.

The *conjugate quaternion* and the *conjugate dual quaternion* of a dual quaternion $Q = x_0 + \varepsilon y_0 + \mathbf{x} + \varepsilon \mathbf{y}$ are defined as

$$\overline{Q} = x_0 + \varepsilon y_0 - \mathbf{x} - \varepsilon \mathbf{y} \quad \text{and} \quad Q_e = x_0 - \varepsilon y_0 + \mathbf{x} - \varepsilon \mathbf{y}, \tag{20}$$

respectively. The norm of a dual quaternion is

$$\|Q\| = \sqrt{Q\overline{Q}}. \tag{21}$$

With these definitions, the equation $\mathbf{b} = \mathbf{X} \cdot \mathbf{a}$ where $\mathbf{X}$ is a matrix of the shape (3) can be written as

$$B = X_e \star A \star \overline{X} \tag{22}$$

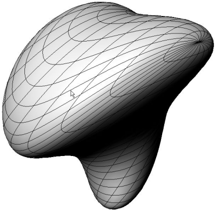
where $X = \mathbf{x} + \varepsilon\mathbf{y}$, $\|X\| = 1$, $\mathbf{x} = (x_0, \dots, x_3)^T$, $\mathbf{y} = (y_0, \dots, y_3)^T$, and $\mathbf{x} \cdot \mathbf{y} = 0$.

The last condition is precisely the Study condition (5).

A and B are dual quaternions of the type: $A = 1 + \varepsilon\mathbf{a}$, $B = 1 + \varepsilon\mathbf{b}$

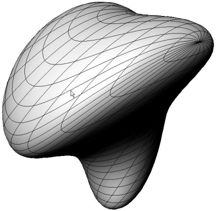
Constraint varieties

$\mapsto$



Constraint varieties

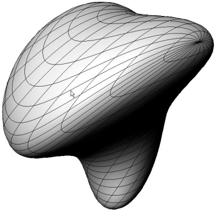
$\mapsto$



- a constraint that removes one degree of freedom maps to a hyper-surface in $\mathbb{P}^7$

Constraint varieties

$\mapsto$



- a constraint that removes one degree of freedom maps to a hyper-surface in $\mathbb{P}^7$
- a set of constraints corresponds to a set of polynomial equations

A simple example

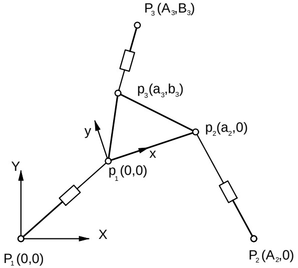


Figure: Planar 3-RPR parallel mechanism.

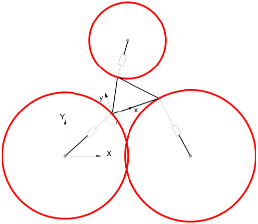


Figure: Geometric interpretations

A simple example

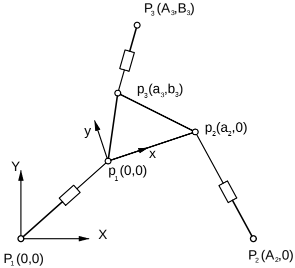


Figure: Planar 3-RPR parallel mechanism.

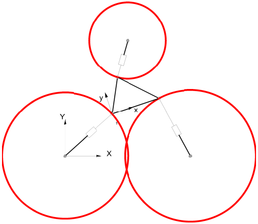


Figure: Geometric interpretations

Condition that one point of the moving system is bound to move on a circle

$$(x_2 - \frac{1}{2}(c_2 + C_2 - x_1(C_1 - c_1)))^2 + (x_3 - \frac{1}{2}(x_1(c_2 - C_2) - C_1 - c_1))^2 - \frac{1}{4}R^2(x_1^2 + 1) = 0, \tag{23}$$

A simple example

Three constraint equations:

$$
\begin{aligned}
h_1 :\quad & 4x_2^2 + 4x_3^2 + R_1 = 0 \\
h_2 :\quad & 4x_2^2 - 4A_2x_3x_0 + 4x_3x_0a_2 + 4x_3^2 - 4x_1x_2a_2 - 4x_1A_2x_2 + 4x_1^2A_2a_2 \\
& -2A_2a_2 + R_2 = 0 \\
h_3 :\quad & 4x_2^2 + 4B_3x_0x_2 - 4A_3x_3x_0 - 4x_2x_0b_3 + 4x_3x_0a_3 + 4x_3^2 - 4x_1B_3x_0a_3 \\
& +4x_1A_3x_0b_3 4x_1x_2a_3 - 4x_1B_3x_3 - 4x_1A_3x_2 - 4x_1x_3b_3 - \\
& +4x_1^2A_3a_3 + 4x_1^2B_3b_3 - 2B_3b_3 - 2A_3a_3 + R_3 = 0. \qquad (24)
\end{aligned}
$$

A simple example

Three constraint equations:

$$
\begin{aligned}
h_1 :\quad & 4x_2^2 + 4x_3^2 + R_1 = 0\\
h_2 :\quad & 4x_2^2 - 4A_2 x_3 x_0 + 4x_3 x_0 a_2 + 4x_3^2 - 4x_1 x_2 a_2 - 4x_1 A_2 x_2 + 4x_1^2 A_2 a_2\\
& -2A_2 a_2 + R_2 = 0\\
h_3 :\quad & 4x_2^2 + 4B_3 x_0 x_2 - 4A_3 x_3 x_0 - 4x_2 x_0 b_3 + 4x_3 x_0 a_3 + 4x_3^2 - 4x_1 B_3 x_0 a_3\\
& +4x_1 A_3 x_0 b_3 4x_1 x_2 a_3 - 4x_1 B_3 x_3 - 4x_1 A_3 x_2 - 4x_1 x_3 b_3 -\\
& +4x_1^2 A_3 a_3 + 4x_1^2 B_3 b_3 - 2B_3 b_3 - 2A_3 a_3 + R_3 = 0.
\end{aligned}
\tag{24}
$$

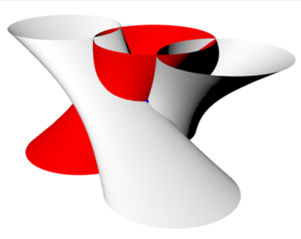


Figure: Geometric interpretation in kinematic image space

Image space transformations

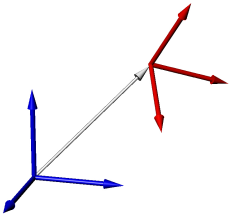


Figure: Fixed and moving coordinate systems

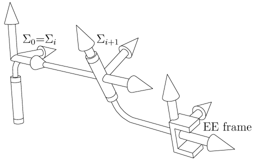


Figure: Robot coordinate systems

Image space transformations

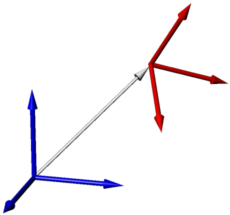


Figure: Fixed and moving coordinate systems

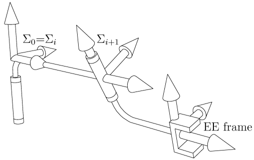


Figure: Robot coordinate systems

- The relative displacement α depends on the choice of fixed and moving frame
- Coordinate systems are usually attached to the base and the end-effector of a mechanism
- Changes of fixed and moving frame induce transformations on S_6^2, impose a geometric structure on S_6^2.

Image space transformations

$$\mathbf{y} = \mathbf{T}_f \mathbf{T}_m \mathbf{x}, \quad \mathbf{T}_m = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \quad \mathbf{T}_f = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{D} & \mathbf{C} \end{bmatrix}, \tag{25}$$

Image space transformations

$$\mathbf{y} = \mathbf{T}_f \mathbf{T}_m \mathbf{x}, \quad \mathbf{T}_m = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \quad \mathbf{T}_f = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{D} & \mathbf{C} \end{bmatrix}, \tag{25}$$

$$\mathbf{A} = \begin{bmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ m_1 & m_0 & m_3 & -m_2 \\ m_2 & -m_3 & m_0 & m_1 \\ m_3 & m_2 & -m_1 & m_0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} m_4 & -m_5 & -m_6 & -m_7 \\ m_5 & m_4 & m_7 & -m_6 \\ m_6 & -m_7 & m_4 & m_5 \\ m_7 & m_6 & -m_5 & m_4 \end{bmatrix} \tag{26}$$

$$\mathbf{C} = \begin{bmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & -f_3 & f_2 \\ f_2 & f_3 & f_0 & -f_1 \\ f_3 & -f_2 & f_1 & f_0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} f_4 & -f_5 & -f_6 & -f_7 \\ f_5 & f_4 & -f_7 & f_6 \\ f_6 & f_7 & f_4 & -f_5 \\ f_7 & -f_6 & f_5 & f_4 \end{bmatrix} \tag{27}$$

and $\mathbf{O}$ is the four by four zero matrix.

- $\mathbf{T}_m$ and $\mathbf{T}_f$ commute
- $\mathbf{T}_m$ and $\mathbf{T}_f$ induce transformations of P^7 that fix S_6^2, the exceptional generator F, and the *exceptional quadric* $E \subset F$

Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials

Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0, \dots x_n]$.

Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0, \dots x_n]$.
- If k is a field and $f_1, \dots, f_s$ are polynomials in $k[x_0, \dots x_n]$, and if

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n : f_i(a_1, \dots, a_n) = 0, \quad \text{for all} \quad 1 \leq i \leq s\}$$

then $\mathbf{V}(f_1, \dots, f_s)$ is called an affine variety defined by the polynomials f_i.

Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0, \dots x_n]$.
- If k is a field and $f_1, \dots, f_s$ are polynomials in $k[x_0, \dots x_n]$, and if

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n : f_i(a_1, \dots, a_n) = 0, \quad \text{for all} \quad 1 \leq i \leq s\}$$

  then $\mathbf{V}(f_1, \dots, f_s)$ is called an affine variety defined by the polynomials f_i.
- The definition says essentially that the affine variety is the zero set of the defining polynomials.

Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0, \dots x_n]$.
- If k is a field and $f_1, \dots, f_s$ are polynomials in $k[x_0, \dots x_n]$, and if

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n : f_i(a_1, \dots, a_n) = 0, \quad \text{for all} \quad 1 \leq i \leq s\}$$

  then $\mathbf{V}(f_1, \dots, f_s)$ is called an affine variety defined by the polynomials f_i.
- The definition says essentially that the affine variety is the zero set of the defining polynomials.
- In case of homogeneous polynomials the variety is called a projective variety.

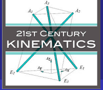
Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0,\dots x_n]$.
- If k is a field and $f_1,\dots,f_s$ are polynomials in $k[x_0,\dots x_n]$, and if

$$\mathbf{V}(f_1,\dots,f_s)=\{(a_1,\dots,a_n)\in k^n : f_i(a_1,\dots,a_n)=0, \quad \text{for all} \quad 1\le i\le s\}$$

  then $\mathbf{V}(f_1,\dots,f_s)$ is called an affine variety defined by the polynomials f_i.
- The definition says essentially that the affine variety is the zero set of the defining polynomials.
- In case of homogeneous polynomials the variety is called a projective variety.
- An ideal I is a subset of $k[x_0,\dots x_n]$ that satisfies the following properties:

$$(i)\ 0\in I.$$
$$(ii)\ \text{If } f,g\in I, \quad \text{then} \quad f+g\in I.$$
$$(iii)\ \text{If } f\in I,\ g\in k \quad \text{then} \quad fg\in I.$$



Affine (Projective) Varieties - Ideals

- A set of constraints is described by a set of polynomials
- The set of polynomials forms a ring which is denoted by $k[x_0,\dots x_n]$.
- If k is a field and $f_1,\dots,f_s$ are polynomials in $k[x_0,\dots x_n]$, and if

$$\mathbf{V}(f_1,\dots,f_s)=\{(a_1,\dots,a_n)\in k^n : f_i(a_1,\dots,a_n)=0, \quad \text{for all} \quad 1\le i\le s\}$$

  then $\mathbf{V}(f_1,\dots,f_s)$ is called an affine variety defined by the polynomials f_i.
- The definition says essentially that the affine variety is the zero set of the defining polynomials.
- In case of homogeneous polynomials the variety is called a projective variety.
- An ideal I is a subset of $k[x_0,\dots x_n]$ that satisfies the following properties:

$$\begin{aligned}
&(i)\ 0\in I.\\
&(ii)\ \text{If } f,g\in I, \quad \text{then} \quad f+g\in I.\\
&(iii)\ \text{If } f\in I,\ g\in k \quad \text{then} \quad fg\in I.
\end{aligned}$$

D. A. Cox, J. B. Little, and D. O'Shea, Ideals, Varieties and Algorithms, Springer, third ed., 2007.

Example: Stewart-Gough platform

Figure: Stewart-Gough platform

Sphere constraint:

1. in canonical form

$$4y_0^2 + 4y_3^2 + 4y_2^2 + 4y_1^2 - (x_1^2 + x_2^2 + x_0^2 + x_3^2)r = 0$$

Sphere constraint:

2 in general form

$$
\begin{aligned}
h\colon\ & R(x_0^2+x_1^2+x_2^2+x_3^2)+4(y_0^2+y_1^2+y_2^2+y_3^2)-2x_0^2(Aa+Bb+Cc)\\
&+2x_1^2(-Aa+Bb+Cc)+2x_2^2(Aa-Bb-Cc)+2x_3^2(Aa+Bb+Cc)\\
&+2x_3^2(Aa+Bb-Cc)+4[x_0x_1(Bc-Cb)+x_0x_2(Ca-Ac)\\
&+x_0x_3(Ab-Ba)-x_1x_2(Ab+Ba)-x_1x_3(Ac+Ca)\\
&-x_2x_3(Bc+Cb)+(x_0y_1-y_0x_1)(A-a)+(x_0y_2-y_0x_2)(B-b)\\
&+(x_0y_3-y_0x_3)(C-c)+(x_1y_2-y_1x_2)(C+c)-(x_1y_3-y_1x_3)(B+b)\\
&+(x_2y_3-y_2x_3)(A+a)]=0,
\end{aligned}
\tag{28}
$$

Sphere constraint:

2 in general form

$$
\begin{aligned}
h\colon\ & R(x_0^2+x_1^2+x_2^2+x_3^2)+4(y_0^2+y_1^2+y_2^2+y_3^2)-2x_0^2(Aa+Bb+Cc)\\
&+2x_1^2(-Aa+Bb+Cc)+2x_2^2(Aa-Bb-Cc)+2x_3^2(Aa+Bb+Cc)\\
&+2x_3^2(Aa+Bb-Cc)+4[x_0x_1(Bc-Cb)+x_0x_2(Ca-Ac)\\
&+x_0x_3(Ab-Ba)-x_1x_2(Ab+Ba)-x_1x_3(Ac+Ca)\\
&-x_2x_3(Bc+Cb)+(x_0y_1-y_0x_1)(A-a)+(x_0y_2-y_0x_2)(B-b)\\
&+(x_0y_3-y_0x_3)(C-c)+(x_1y_2-y_1x_2)(C+c)-(x_1y_3-y_1x_3)(B+b)\\
&+(x_2y_3-y_2x_3)(A+a)]=0, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
F:=[&177x_2y_3-177x_3y_2-20x_1y_0+20x_0y_1-34059x_0x_3+12236x_2x_1-x_0^2S_1-x_1^2S_1-x_3^2S_1-x_2^2S_1,\\
&156x_2y_3-156x_3y_2-101x_1y_0+101x_0y_1+68081x_0x_3-101796x_2x_1-x_0^2S_2-x_1^2S_2-x_3^2S_2-x_2^2S_2,\\
&-x_0^2S_3-x_1^2S_3-x_3^2S_3-x_2^2S_3-198x_2y_3+198x_3y_2-61x_1y_0+61x_0y_1-68203x_0x_3-126565x_2x_1,\\
&438313x_2^2+x_0^2S_4+x_1^2S_4+x_3^2S_4+x_2^2S_4+792x_2y_3-792x_3y_2+244x_1y_0-244x_0y_1-1370x_3y_1+\\
&422x_0y_2-422x_2y_0+1370y_3x_1-544796x_0x_3+505072x_2x_1-437869x_1^2-11x_0^2+455x_3^2,\\
&-438313x_2^2-x_0^2S_5-x_1^2S_5-x_3^2S_5-x_2^2S_5+792x_2y_3-792x_3y_2+244x_1y_0-244x_0y_1+1370x_3y_1\\
&-422x_0y_2+422x_2y_0-1370y_3x_1-544796x_0x_3+505072x_2x_1+437869x_1^2+11x_0^2-455x_3^2,\\
&x_0y_0+x_1y_1+x_2y_2+x_3y_3,\\
&-x_0^2W_1-x_1^2W_1-x_3^2W_1-x_2^2W_1-204402x_0x_3-297x_2x_1]
\end{aligned}
$$

40 solutions, H. (1996)

Implicitization Algorithm

Is there a method to generate constraint equations without (deep) insight in the geometric structure of a kinematic chain??

Implicitization Algorithm

Is there a method to generate constraint equations without (deep) insight in the geometric structure of a kinematic chain??

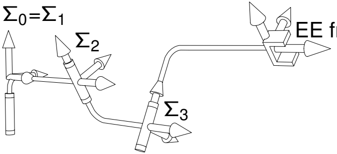


Figure: Canonical 3R-chain

the relative position of two rotation axes is described by the usual Denavit-Hartenberg parameters (α_i, a_i, d_i)

$$\mathbf{G}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_i & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ d_i & 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}. \tag{29}$$

$$\mathbf{M}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(u_i) & -\sin(u_i) & 0 \\ 0 & \sin(u_i) & \cos(u_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \mathbf{M}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

Following this sequence of transformations the endeffector will have the following pose:

$$\mathbf{D} = \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \dots \cdot \mathbf{M}_n, \tag{31}$$

Parametric equation

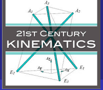
parametric $\to$ implicit

What do we gain?

parametric $\to$ implicit

What do we gain?

- Using all features of algebraic geometry symbolic software (Maple, Mathematica, Singular, ...) e.g.:



parametric → implicit

What do we gain?

- Using all features of algebraic geometry symbolic software (Maple, Mathematica, Singular, ...) e.g.:

  with(PolynomialIdeals):

  [`<,>`, *Add, Contract, EliminationIdeal, EquidimensionalDecomposition, Generators, HilbertDimension, IdealContainment, IdealInfo, IdealMembership, Intersect, IsMaximal, IsPrimary, IsPrime, IsProper, IsRadical, IsZeroDimensional, MaximalIndependentSet, Multiply, NumberOfSolutions, Operators, PolynomialIdeal, PrimaryDecomposition, PrimeDecomposition, Quotient, Radical, RadicalMembership, Saturate, Simplify, UnivariatePolynomial, VanishingIdeal, ZeroDimensionalDecomposition, in, subset*]

  with(Groebner);

  [*Basis, FGLM, HilbertDimension, HilbertPolynomial, HilbertSeries, Homogenize, InitialForm, InterReduce, IsProper, IsZeroDimensional, LeadingCoefficient, LeadingMonomial, LeadingTerm, MatrixOrder, MaximalIndependentSet, MonomialOrder, MultiplicationMatrix, MultivariateCyclicVector, NormalForm, NormalSet, RationalUnivariateRepresentation, Reduce, RememberBasis, SPolynomial, Solve, SuggestVariableOrder, TestOrder, ToricIdealBasis, TrailingTerm, UnivariatePolynomial, Walk, WeightedDegree*]

parametric → implicit

What do we gain?

- Using all features of algebraic geometry symbolic software (Maple, Mathematica, Singular, ...) e.g.:

  with(PolynomialIdeals):

  ['<,>',Add,Contract,EliminationIdeal, EquidimensionalDecomposition, Generators, HilbertDimension, IdealContainment, IdealInfo, IdealMembership, Intersect,IsMaximal, IsPrimary, IsPrime, IsProper, IsRadical, IsZeroDimensional, MaximalIndependentSet, Multiply, NumberOfSolutions, Operators, PolynomialIdeal, PrimaryDecomposition, PrimeDecomposition, Quotient, Radical, RadicalMembership, Saturate, Simplify, UnivariatePolynomial, VanishingIdeal, ZeroDimensionalDecomposition, in, subset]

  with(Groebner);

  [Basis,FGLM, HilbertDimension, HilbertPolynomial ,HilbertSeries, Homogenize, InitialForm, InterReduce,IsProper, IsZeroDimensional,LeadingCoefficient,LeadingMonomial, LeadingTerm, MatrixOrder, MaximalIndependentSet, MonomialOrder, MultiplicationMatrix, MultivariateCyclicVector, NormalForm, NormalSet, RationalUnivariateRepresentation, Reduce, RememberBasis,SPolynomial,Solve,SuggestVariableOrder, TestOrder, ToricIdealBasis,TrailingTerm, UnivariatePolynomial, Walk, WeightedDegree]

- all solutions, sometimes a complete analytic description of a workspace.

parametric → implicit

What do we gain?

- Using all features of algebraic geometry symbolic software (Maple, Mathematica, Singular, ...) e.g.:

  with(PolynomialIdeals):

  [`<,>`, *Add*, *Contract*, *EliminationIdeal*, *EquidimensionalDecomposition*, *Generators*, *HilbertDimension*, *IdealContainment*, *IdealInfo*, *IdealMembership*, *Intersect*, *IsMaximal*, *IsPrimary*, *IsPrime*, *IsProper*, *IsRadical*, *IsZeroDimensional*, *MaximalIndependentSet*, *Multiply*, *NumberOfSolutions*, *Operators*, *PolynomialIdeal*, *PrimaryDecomposition*, *PrimeDecomposition*, *Quotient*, *Radical*, *RadicalMembership*, *Saturate*, *Simplify*, *UnivariatePolynomial*, *VanishingIdeal*, *ZeroDimensionalDecomposition*, *in*, *subset*]

  with(Groebner);

  [*Basis*, *FGLM*, *HilbertDimension*, *HilbertPolynomial* , *HilbertSeries*, *Homogenize*, *InitialForm*, *InterReduce*, *IsProper*, *IsZeroDimensional*, *LeadingCoefficient*, *LeadingMonomial*, *LeadingTerm*, *MatrixOrder*, *MaximalIndependentSet*, *MonomialOrder*, *MultiplicationMatrix*, *MultivariateCyclicVector*, *NormalForm*, *NormalSet*, *RationalUnivariateRepresentation*, *Reduce*, *RememberBasis*, *SPolynomial*, *Solve*, *SuggestVariableOrder*, *TestOrder*, *ToricIdealBasis*, *TrailingTerm*, *UnivariatePolynomial*, *Walk*, *WeightedDegree*]
- all solutions, sometimes a complete analytic description of a workspace.
- Singularities can be treated, pathologic cases (selfmotion) can be detected and degree of freedom computation (Hilbert dimension) can be performed

Implicitization Algorithm

Back to the parametric equations!

Implicitization Algorithm

Back to the parametric equations!

Half tangent substitution transforms the rotation angles u_i into algebraic parameters t_i and one ends up with eight parametric equations of the form:

$$\begin{aligned} x_0 &= f_0(t_1, \dots t_n), \\ x_1 &= f_1(t_1, \dots t_n), \\ &\vdots \\ y_3 &= f_8(t_1, \dots t_n). \end{aligned} \tag{32}$$

- Equations will be rational having a denominator of the form $(1 + t_1^2) \cdot \ldots \cdot (1 + t_n^2)$ which can be canceled because the Study parameters x_i, y_i are homogeneous.
- The same can be done with a possibly appearing common factor of all parametric expressions.

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \ldots, t_n$

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
  - $\rightarrow$ kinematic mapping a set of corresponding points in P^7

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
    - $\rightarrow$ kinematic mapping a set of corresponding points in P^7
    - ask now for the smallest variety $\mathscr{V} \in P^7$ (with respect to inclusion) which contains all these points

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
  - $\rightarrow$ kinematic mapping a set of corresponding points in P^7
  - ask now for the smallest variety $\mathscr{V} \in P^7$ (with respect to inclusion) which contains all these points
- What do we know about this variety?

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
  - $\rightarrow$ kinematic mapping a set of corresponding points in P^7
  - ask now for the smallest variety $\mathscr{V} \in P^7$ (with respect to inclusion) which contains all these points
- What do we know about this variety?
- Its ideal $\mathscr{V}$ consists of homogeneous polynomials and contains $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, i.e. the equation for the Study quadric S_6^2.

Implicitization Algorithm

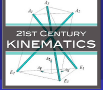
- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
  - $\rightarrow$ kinematic mapping a set of corresponding points in P^7
  - ask now for the smallest variety $\mathscr{V} \in P^7$ (with respect to inclusion) which contains all these points
- What do we know about this variety?
- Its ideal $\mathscr{V}$ consists of homogeneous polynomials and contains $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, i.e. the equation for the Study quadric S_6^2.
- the minimum number of polynomials to describe $\mathscr{V}$ corresponds to the degrees of freedom (dof) of the kinematic chain

Implicitization Algorithm

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric
- transformation parametrized by n parameters $t_1, \dots, t_n$
  - $\rightarrow$ kinematic mapping a set of corresponding points in P^7
  - ask now for the smallest variety $\mathscr{V} \in P^7$ (with respect to inclusion) which contains all these points
- What do we know about this variety?
- Its ideal $\mathscr{V}$ consists of homogeneous polynomials and contains $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, i.e. the equation for the Study quadric S_6^2.
- the minimum number of polynomials to describe $\mathscr{V}$ corresponds to the degrees of freedom (dof) of the kinematic chain
- If the number of generic parameters is n then $m = 6 - n$ polynomials are necessary to describe $\mathscr{V}$

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations



Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

Manfred L. Husty, Hans-Peter Schröcker

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p

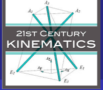
Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i



Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all t_i $\rightarrow$

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all $t_i \rightarrow$
  - all coefficients have to vanish $\rightarrow$

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all $t_i \rightarrow$
  - all coefficients have to vanish $\rightarrow$
  - collect with respect to the powerproducts of the t_i and extract their coefficients $\rightarrow$

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all t_i $\rightarrow$
  - all coefficients have to vanish $\rightarrow$
  - collect with respect to the powerproducts of the t_i and extract their coefficients $\rightarrow$
  - system of linear equations in the $\binom{n+7}{n}$ coefficients C_k

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all t_i $\rightarrow$
  - all coefficients have to vanish $\rightarrow$
  - collect with respect to the powerproducts of the t_i and extract their coefficients $\rightarrow$
  - system of linear equations in the $\binom{n+7}{n}$ coefficients C_k
- determine C_k

Implicitization Algorithm

General observation: the parametric equations of a geometric object have to fulfill the implicit equations

- Make a general ansatz of a polynomial of degree n:

$$p = \sum_{\alpha,\beta} C_k x_i^{\alpha} y_j^{\beta}$$

- substitute the parametric equations into p
  - resulting expression is a polynomial f in t_i
  - f has to vanish for all t_i $\rightarrow$
  - all coefficients have to vanish $\rightarrow$
  - collect with respect to the powerproducts of the t_i and extract their coefficients $\rightarrow$
  - system of linear equations in the $\binom{n+7}{n}$ coefficients C_k
- determine C_k
- possibly increase the degree of the ansatz polynomial

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i
- system is highly overconstrained

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i
- system is highly overconstrained
- equations have to be dependent, at least if the degree of the ansatz polynomial is increased, because the constraint variety will have some algebraic degree.

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i
- system is highly overconstrained
- equations have to be dependent, at least if the degree of the ansatz polynomial is increased, because the constraint variety will have some algebraic degree.
- if these systems can be solved depends how complicated the chain is (we have solved up to degree 8)

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i
- system is highly overconstrained
- equations have to be dependent, at least if the degree of the ansatz polynomial is increased, because the constraint variety will have some algebraic degree.
- if these systems can be solved depends how complicated the chain is (we have solved up to degree 8)
- in a step of the algorithm polynomials could be created that are contained in the ideal of polynomials created in steps before. Test and reduce w.r.t. a Grö bner basis

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Remarks:

- The number of equations depends on the particular design of the chain
- in general the system will consist of more equations than unknowns because in general there are more powerproducts than unknowns C_i
- system is highly overconstrained
- equations have to be dependent, at least if the degree of the ansatz polynomial is increased, because the constraint variety will have some algebraic degree.
- if these systems can be solved depends how complicated the chain is (we have solved up to degree 8)
- in a step of the algorithm polynomials could be created that are contained in the ideal of polynomials created in steps before. Test and reduce w.r.t. a Grö bner basis
- the algorithm could create more polynomials than needed; take out of the set the number needed (simplest!)

Walter and H. , On Implicitization of Kinematic Constraint Equations, Chin. J. of Mech. Design, 2010.

Constraint equations and mechanism freedom

Definition

The degree of freedom of a mechanical system is the Hilbert dimension of the ideal generated by the constraint polynomials, the Study quadric and a normalizing condition

Constraint equations and mechanism freedom

Definition

The degree of freedom of a mechanical system is the Hilbert dimension of the ideal generated by the constraint polynomials, the Study quadric and a normalizing condition

Example: Self motions of Stewart Platforms

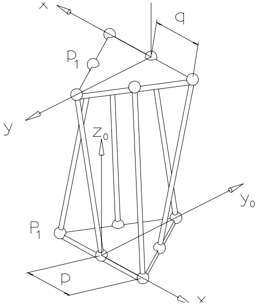
```
> with(Groebner):
> F:=[U4,U2,U3,U8,U10,U7,h1,x0^2+x1^2+x2^2+x3^2-1];
```

$$
\begin{aligned}
F := [&244x_1y_0 - 792x_3y_2 - 244x_0y_1 + 1370y_3x_1 - 1370x_3y_1 + 422x_0y_2 + 439323x_2^2 + 1465x_3^2 + 999x_0^2 - \\
&436859x_1^2 + 792x_2y_3 - 422x_2y_0 - 544796x_0x_3 + 505072x_2x_1,\ -101x_1y_0 - 156x_3y_2 + 101x_0y_1 + 156x_2y_3 + \\
&68081x_0x_3 - 101796x_2x_1 - \frac{4401}{4}x_1^2 - \frac{4401}{4}x_3^2 - \frac{4401}{4}x_2^2 - \frac{4401}{4}x_0^2, -61x_1y_0 + 198x_3y_2 + 61x_0y_1 - 198x_2y_3 - \\
&68203x_0x_3 - 126565x_2x_1 - \frac{6713}{2}x_1^2 - \frac{6713}{2}x_3^2 - \frac{6713}{2}x_2^2 - \frac{6713}{2}x_0^2, -204402x_0x_3 - 297x_2x_1 - \frac{3749}{2}x_1^2 - \\
&\frac{3749}{2}x_3^2 - \frac{3749}{2}x_2^2 - \frac{3749}{2}x_0^2, -404x_1y_0 - 624x_3y_2 + 404x_0y_1 + 1082y_3x_1 - 1082x_3y_1 - 700x_0y_2 - 375644x_2^2 - \\
&22627x_3^2 - 22712x_0^2 + 330305x_1^2 + 624x_2y_3 + 700x_2y_0 - 545284x_0x_3 - 408372x_2x_1,\ x_0y_0 + x_1y_1 + x_2y_2 + \\
&x_3y_3, -640x_1y_0 - 5664x_3y_2 + 640x_0y_1 + 384y_3x_1 - 384x_3y_1 + 1496x_0y_2 + 4y_0^2 + 4y_3^2 + 4y_2^2 + 4y_1^2 + 1891923x_2^2 + \\
&1761263x_3^2 - 87533x_0^2 - 218193x_1^2 + 5664x_2y_3 - 1496x_2y_0 - 1089888x_0x_3 + 391552x_2x_1,\ x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1]
\end{aligned}
$$

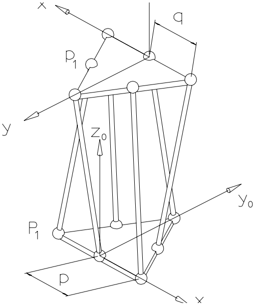
```
> HilbertDimension(F,tdeg(x0,x1,x2,x3,y0,y1,y2,y3));
```

0

Griffis-Duffy platform



Griffis-Duffy platform




```
> with(Groebner):
> G:=[U2,U3,U4,U5,h1,U7,x0^2+x1^2+x2^2+x3^2-1];
```

$G := [-4x_2y_3 + 12x_1y_0 + 4y_2x_3 + 4\sqrt{3}x_1x_2, 8\sqrt{3}x_2y_0, -4x_2y_3 - 12x_1y_0 + 4y_2x_3 + 4\sqrt{3}x_1x_2, -\frac{2}{3}\sqrt{3}(\sqrt{3}x_2y_3 - 3\sqrt{3}x_1y_0 - \sqrt{3}y_2x_3 + \sqrt{3}x_2^2 + \sqrt{3}x_3^2 - 3x_1x_2 + 3x_2y_0 - 3y_3x_1 + 3x_3y_1), 4\sqrt{3}y_0(\sqrt{3}x_1 + x_2), 4y_0^2 + 4y_1^2 + 4y_3^2 + 4y_2^2 + x_3^2(2-R) + x_1^2(2-R) + x_2^2(2-R) + 2\sqrt{3}x_3y_1 + 6x_1y_0 + 2y_2x_3 + 2\sqrt{3}x_2y_0 - 2x_2y_3 - 2\sqrt{3}y_3x_1 + x_1^2 - x_3^2 + 2\sqrt{3}x_1x_2 - x_2^2, x_1y_1 + x_2y_2 + x_3y_3, -1 + x_1^2 + x_2^2 + x_3^2]$

```
> HilbertDimension(F,tdeg(x1,x2,x3,y0,y1,y2,y3));
```

1

Schatz Mechanism - Bricard's overconstrained 6R chain

Turbula T2F Heavy-Duty Shaker-Mixer (Willy A. Bachofen AG, http://www.wab.ch/ie/e/turbula1.htm

The DH parameters of Bricard's orthogonal chain

i	a_i	d_i	α_i
1	a_1	0	$\pi/2$
2	a_2	0	$\pi/2$
3	a_3	0	$\pi/2$
4	a_4	0	$\pi/2$
5	a_5	0	$\pi/2$
6	a_6	0	$\pm\pi/2$

Table: DH parameters of Bricard's orthogonal chain

with the additional condition that $a_1^2 - a_2^2 + a_3^2 - a_4^2 + a_5^2 - a_6^2 = 0$.

Schatz Mechanism - Bricard's overconstrained 6R chain

$$F := [-2z_0 + z_1 + z_2 v_1 - 2z_3 v_1 - 2s_1 - 2s_2 v_1, -z_0 + 2z_1 + 2z_2 v_1 - z_3 v_1 - 2s_0 - 2s_3 v_1, -z_1 v_1 + z_2 - 2s_1 v_1 + 2s_2, -z_0 v_1 + z_3 + 2s_0 v_1 - 2s_3, z_0 v_{6q} + z_1 + 2z_1 v_{6q} - z_2 + 2z_2 v_{6q} + z_3 v_{6q} - 2s_0 v_{6q} - 2s_1 + 2s_2 - 2s_3 v_{6q}, -z_0 + 2z_0 v_{6q} + z_1 v_{6q} + z_2 v_{6q} + z_3 + 2z_3 v_{6q} - 2s_0 + 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, z_0 + 2z_0 v_{6q} - z_1 v_{6q} + z_2 v_{6q} + z_3 - 2z_3 v_{6q} + 2s_0 - 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, -z_0 v_{6q} - z_1 + 2z_1 v_{6q} - z_2 - 2z_2 v_{6q} + z_3 v_{6q} + 2s_0 v_{6q} + 2s_1 + 2s_2 - 2s_3 v_{6q}, s_0 z_0 + s_1 z_1 + s_2 z_2 + s_3 z_3, z_0^2 + z_1^2 + z_2^2 + z_3^2 - 1]$$

> *HilbertDimension*(F);

1

Schatz Mechanism - Bricard's overconstrained 6R chain

$$F := [-2z_0 + z_1 + z_2 v_1 - 2z_3 v_1 - 2s_1 - 2s_2 v_1, -z_0 + 2z_1 + 2z_2 v_1 - z_3 v_1 - 2s_0 - 2s_3 v_1, -z_1 v_1 + z_2 - 2s_1 v_1 + 2s_2, -z_0 v_1 + z_3 + 2s_0 v_1 - 2s_3, z_0 v_{6q} + z_1 + 2z_1 v_{6q} - z_2 + 2z_2 v_{6q} + z_3 v_{6q} - 2s_0 v_{6q} - 2s_1 + 2s_2 - 2s_3 v_{6q}, -z_0 + 2z_0 v_{6q} + z_1 v_{6q} + z_2 v_{6q} + z_3 + 2z_3 v_{6q} - 2s_0 + 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, z_0 + 2z_0 v_{6q} - z_1 v_{6q} + z_2 v_{6q} + z_3 - 2z_3 v_{6q} + 2s_0 - 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, -z_0 v_{6q} - z_1 + 2z_1 v_{6q} - z_2 - 2z_2 v_{6q} + z_3 v_{6q} + 2s_0 v_{6q} + 2s_1 + 2s_2 - 2s_3 v_{6q}, s_0 z_0 + s_1 z_1 + s_2 z_2 + s_3 z_3, z_0^2 + z_1^2 + z_2^2 + z_3^2 - 1]$$

> *HilbertDimension(F);*

1

```
> Basis(F,tdeg[z0,z1,z2,z3,s0,s1,s2,s2,v6q,v1]);
```

$$F := [z_2 - z_1 - z_0 + z_3, 2s_1 - 2s_0 + z_1 + z_0, 2s_2 + 2s_0 - z_1 + z_3, 2s_3 + 2s_0 - 2z_1 - z_0 + z_3, 2s_0 z_0 - 2s_0 z_3 + 2z_1 z_3 + z_0 z_3 - z_3^2, z_0 v_1 - z_3 v_1 - 2s_0 + 2z_1 - z_3, 2z_1^2 - 1 + 2z_1 z_0 + 2z_0^2 - 2z_1 z_3 - 2z_0 z_3 + 2z_3^2, z_1 v_{6q} + z_0 v_{6q} - 2s_0 + z_1, 4s_0^2 - 4s_0 z_1 - z_0^2 + 2z_1 z_3 + 2z_0 z_3 - 2z_3^2, 2s_0 v_1 - z_3 v_1 - z_0 + z_3, 2s_0 v_{6q} - z_0 v_{6q} + 2z_3 v_{6q} - 2s_0 - z_0, 8v_{6q} z_3^2 - 2 - 4v_1 z_3^2 + v_{6q} v_1 + 12s_0 z_1 + 2z_1 z_0 + 2z_0^2 - 12s_0 z_3 - 10z_0 z_3 + 6z_3^2 - 4v_{6q} + v_1, 8v_{6q} z_0 z_3 - 1 + 4s_0 z_1 + 2z_1 z_0 - 12s_0 z_3 + 8z_1 z_3 + 2z_0 z_3 - v_{6q}, 2v_1 z_1 z_3 + z_0^2 + 4s_0 z_3 - 4z_1 z_3 - 2z_0 z_3 + 3z_3^2, v_{6q} v_1 z_3 - z_0 v_{6q} - z_3 v_1 - z_0, z_0^3 + 4s_0 z_1 z_3 + z_0^2 z_3 - 2z_1 z_3^2 - z_0 z_3^2 + 3z_3^3 - 2z_3, 8v_{6q} z_0^2 - 2 + 4v_1 z_3^2 - v_{6q} v_1 + 4s_0 z_1 + 6z_1 z_0 + 6z_0^2 - 4s_0 z_3 + 2z_0 z_3 + 2z_3^2 - v_1, 2v_1 z_3^3 - z_1 z_0^2 - 2z_1 z_0 z_3 + 4s_0 z_3^2 - 5z_1 z_3^2 + 2z_3^3 - z_3 v_1, 4v_1^2 z_3^2 + 1 - v_{6q} v_1^2 + 4v_{6q} v_1 - v_1^2 - 4z_1 z_0 + 16s_0 z_3 - 12z_1 z_3 - 4z_0 z_3 + 8z_3^2 - v_{6q} + 2v_1, \color{red}{v_{6q}^2 v_1^2 - 1 - 4v_{6q}^2 v_1 + v_{6q}^2 - v_1^2}]$$

Schatz Mechanism - Bricard's overconstrained 6R chain

$$F := [-2z_0 + z_1 + z_2 v_1 - 2z_3 v_1 - 2s_1 - 2s_2 v_1, -z_0 + 2z_1 + 2z_2 v_1 - z_3 v_1 - 2s_0 - 2s_3 v_1, -z_1 v_1 + z_2 - 2s_1 v_1 + 2s_2, -z_0 v_1 + z_3 + 2s_0 v_1 - 2s_3, z_0 v_{6q} + z_1 + 2z_1 v_{6q} - z_2 + 2z_2 v_{6q} + z_3 v_{6q} - 2s_0 v_{6q} - 2s_1 + 2s_2 - 2s_3 v_{6q}, -z_0 + 2z_0 v_{6q} + z_1 v_{6q} + z_2 v_{6q} + z_3 + 2z_3 v_{6q} - 2s_0 + 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, z_0 + 2z_0 v_{6q} - z_1 v_{6q} + z_2 v_{6q} + z_3 - 2z_3 v_{6q} + 2s_0 - 2s_1 v_{6q} + 2s_2 v_{6q} + 2s_3, -z_0 v_{6q} - z_1 + 2z_1 v_{6q} - z_2 - 2z_2 v_{6q} + z_3 v_{6q} + 2s_0 v_{6q} + 2s_1 + 2s_2 - 2s_3 v_{6q}, s_0 z_0 + s_1 z_1 + s_2 z_2 + s_3 z_3, z_0^2 + z_1^2 + z_2^2 + z_3^2 - 1]$$

> $HilbertDimension(F);$

$$1$$

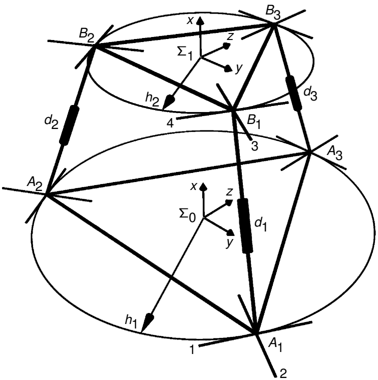
```
> Basis(F,tdeg[z0,z1,z2,z3,s0,s1,s2,s2,v6q,v1]);
```

$$F := [z_2 - z_1 - z_0 + z_3, 2s_1 - 2s_0 + z_1 + z_0, 2s_2 + 2s_0 - z_1 + z_3, 2s_3 + 2s_0 - 2z_1 - z_0 + z_3, 2s_0 z_0 - 2s_0 z_3 + 2z_1 z_3 + z_0 z_3 - z_3^2, z_0 v_1 - z_3 v_1 - 2s_0 + 2z_1 - z_3, 2z_1^2 - 1 + 2z_1 z_0 + 2z_0^2 - 2z_1 z_3 - 2z_0 z_3 + 2z_3^2, z_1 v_{6q} + z_0 v_{6q} - 2s_0 + z_1, 4s_0^2 - 4s_0 z_1 - z_0^2 + 2z_1 z_3 + 2z_0 z_3 - 2z_3^2, 2s_0 v_1 - z_3 v_1 - z_0 + z_3, 2s_0 v_{6q} - z_0 v_{6q} + 2z_3 v_{6q} - 2s_0 - z_0, 8v_{6q} z_3^2 - 2 - 4v_1 z_3^2 + v_{6q} v_1 + 12s_0 z_1 + 2z_1 z_0 + 2z_0^2 - 12s_0 z_3 - 10z_0 z_3 + 6z_3^2 - 4v_{6q} + v_1, 8v_{6q} z_0 z_3 - 1 + 4s_0 z_1 + 2z_1 z_0 - 12s_0 z_3 + 8z_1 z_3 + 2z_0 z_3 - v_{6q}, 2v_1 z_1 z_3 + z_0^2 + 4s_0 z_3 - 4z_1 z_3 - 2z_0 z_3 + 3z_3^2, v_{6q} v_1 z_3 - z_0 v_{6q} - z_3 v_1 - z_0, z_0^3 + 4s_0 z_1 z_3 + z_0^2 z_3 - 2z_1 z_3^2 - z_0 z_3^2 + 3z_3^3 - 2z_3, 8v_{6q} z_0^2 - 2 + 4v_1 z_3^2 - v_{6q} v_1 + 4s_0 z_1 + 6z_1 z_0 + 6z_0^2 - 4s_0 z_3 + 2z_0 z_3 + 2z_3^2 - v_1, 2v_1 z_3^3 - z_1 z_0^2 - 2z_1 z_0 z_3 + 4s_0 z_3^2 - 5z_1 z_3^2 + 2z_3^3 - z_3 v_1, 4v_1^2 z_3^2 + 1 - v_{6q} v_1^2 + 4v_{6q} v_1 - v_1^2 - 4z_1 z_0 + 16s_0 z_3 - 12z_1 z_3 - 4z_0 z_3 + 8z_3^2 - v_{6q} + 2v_1, \color{red}{v_{6q}^2 v_1^2 - 1 - 4v_{6q}^2 v_1 + v_{6q}^2 - v_1^2}]$$

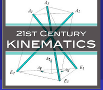
M. Pfurner, PhD thesis, Innsbruck, 2007

http://repository.uibk.ac.at/viewer.alo?viewmode=overview&objid=1015078&page=

The TSAI-UPU Parallel Manipulator



- difference to the SNU-3UPU manipulator: legs are rotated by 90 degrees before assembly



Kinematics and Algebraic Geometry

Manfred L. Husty, Hans-Peter Schröcker

Introduction

Kinematic mapping

Quaternions

Algebraic Geometry and Kinematics

Methods to establish the sets of equations – the canonical equations

Constraint equations and mechanism freedom

The TSAI-UPU Parallel Manipulator

Solving the system of equations

Operation modes

Singular poses

Changing operation modes

Synthesis of mechanisms

The algebraic constraint equations

$$g_1 : x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

$$g_2 : (h_1 - h_2) x_0 x_2 + (h_1 + h_2) x_1 x_3 - x_2 y_3 - x_3 y_2 = 0$$
$$g_3 : (h_1 - h_2) x_0 x_3 - (h_1 + h_2) x_1 x_2 - 4 x_1 y_1 - 3 x_2 y_2 - x_3 y_3 = 0$$
$$g_4 : (h_1 - h_2) x_0 x_3 - (h_1 + h_2) x_1 x_2 + 2 x_1 y_1 + 2 x_3 y_3 = 0$$

$$\begin{aligned} g_5 :& (h_1^2 - 2 h_1 h_2 + h_2^2 - d_1^2) x_0^2 + 2\sqrt{3}(h_1 - h_2) x_0 y_2 - 2(h_1 - h_2) x_0 y_3 + (h_1^2 + 2 h_1 h_2 + h_2^2 - d_1^2) x_1^2 - \\ & - 2(h_1 + h_2) x_1 y_2 - 2\sqrt{3}(h_1 + h_2) x_1 y_3 + (h_1^2 - h_1 h_2 + h_2^2 - d_1^2) x_2^2 + 2\sqrt{3} h_1 h_2 x_2 x_3 - \\ & - 2\sqrt{3}(h_1 - h_2) x_2 y_0 + 2(h_1 + h_2) x_2 y_1 + (h_1^2 + h_1 h_2 + h_2^2 - d_1^2) x_3^2 + 2(h_1 - h_2) x_3 y_0 + \\ & + 2\sqrt{3}(h_1 + h_2) x_3 y_1 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \end{aligned}$$

$$\begin{aligned} g_6 :& (h_1^2 - 2 h_1 h_2 + h_2^2 - d_2^2) x_0^2 - 2\sqrt{3}(h_1 - h_2) x_0 y_2 - 2(h_1 - h_2) x_0 y_3 + (h_1^2 + 2 h_1 h_2 + h_2^2 - d_2^2) x_1^2 - \\ & - 2(h_1 + h_2) x_1 y_2 + 2\sqrt{3}(h_1 + h_2) x_1 y_3 + + (h_1^2 - h_1 h_2 + h_2^2 - d_2^2) x_2^2 - 2\sqrt{3} h_1 h_2 x_2 x_3 + \\ & + 2\sqrt{3}(h_1 - h_2) x_2 y_0 + 2(h_1 + h_2) x_2 y_1 + + (h_1^2 + h_1 h_2 + h_2^2 - d_2^2) x_3^2 + 2(h_1 - h_2) x_3 y_0 - \\ & - 2\sqrt{3}(h_1 + h_2) x_3 y_1 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \end{aligned}$$

$$\begin{aligned} g_7 :& (h_1^2 - 2 h_1 h_2 + h_2^2 - d_3^2) x_0^2 + 4(h_1 - h_2) x_0 y_3 + (h_1^2 + 2 h_1 h_2 + h_2^2 - d_3^2) x_1^2 + 4(h_1 + h_2) x_1 y_2 + \\ & + (h_1^2 + 2 h_1 h_2 + h_2^2 - d_3^2) x_2^2 - 4(h_1 + h_2) x_2 y_1 + (h_1^2 - 2 h_1 h_2 + h_2^2 - d_3^2) x_3^2 - 4(h_1 - h_2) x_3 y_0 + \\ & + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \end{aligned}$$

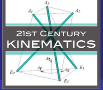
normalization equation is added:

$$g_8 : x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0 \tag{33}$$

Solving the system of equations

- polynomial ideal over the ring $\mathbb{R}[h_1, h_2, d_1, d_2, d_3][x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]$

$$\mathscr{I} = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle$$



Solving the system of equations

- polynomial ideal over the ring $\mathbb{R}[h_1,h_2,d_1,d_2,d_3][x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3]$

$$\mathscr{I} = \langle g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8\rangle$$

- primary decomposition

$$\langle g_1,g_2,g_3,g_4\rangle = \bigcap_{i=1}^{6} \mathscr{J}_i$$

$$\begin{aligned}
&\mathscr{J}_1 = \langle y_0,x_1,x_2,x_3\rangle,\ \mathscr{J}_2 = \langle x_0,y_1,x_2,x_3\rangle, \mathscr{J}_3 = \langle y_0,y_1,x_2,x_3\rangle,\ \mathscr{J}_4 = \langle x_0,x_1,y_2,y_3\rangle,\\
&\mathscr{J}_5 = \langle (h_1-h_2)\,x_0\,x_2 + (h_1+h_2)\,x_1\,x_3 - x_2\,y_3 - x_3\,y_2,\\
&(h_1-h_2)\,x_0\,x_3 - (h_1+h_2)\,x_1\,x_2 - x_2\,y_2 + x_3\,y_3,\\
&2\,x_1\,y_1 + x_2\,y_2 + x_3\,y_3, x_0\,y_0 - x_1\,y_1, (h_1-h_2)^2\,x_0^2 + (h_1+h_2)^2\,x_1^2 - y_2^2 - y_3^2,\\
&(h_1+h_2)\,x_2^3\,y_0 - 3\,(h_1-h_2)\,x_2^2\,x_3\,y_1 - 2\,x_2^2\,y_0\,y_1 -\\
&-3\,(h_1-h_2)\,x_2\,x_3^2\,y_0 + (h_1-h_2)\,x_3^3\,y_1 - 2\,x_3^2\,y_0\,y_1\rangle\\
&\mathscr{J}_6 = \langle x_0,x_1,x_2,x_3\rangle.
\end{aligned}$$

Solving the system of equations

- polynomial ideal over the ring $\mathbb{R}[h_1,h_2,d_1,d_2,d_3][x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3]$

$$\mathscr{I}=\langle g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8\rangle$$

- primary decomposition

$$\langle g_1,g_2,g_3,g_4\rangle=\bigcap_{i=1}^{6}\mathscr{J}_i$$

$$\begin{gathered}
\mathscr{J}_1=\langle y_0,x_1,x_2,x_3\rangle,\ \mathscr{J}_2=\langle x_0,y_1,x_2,x_3\rangle,\ \mathscr{J}_3=\langle y_0,y_1,x_2,x_3\rangle,\ \mathscr{J}_4=\langle x_0,x_1,y_2,y_3\rangle,\\
\mathscr{J}_5=\langle (h_1-h_2)\,x_0\,x_2+(h_1+h_2)\,x_1\,x_3-x_2\,y_3-x_3\,y_2,\\
(h_1-h_2)\,x_0\,x_3-(h_1+h_2)\,x_1\,x_2-x_2\,y_2+x_3\,y_3,\\
2\,x_1\,y_1+x_2\,y_2+x_3\,y_3,x_0\,y_0-x_1\,y_1,(h_1-h_2)^2\,x_0^2+(h_1+h_2)^2\,x_1^2-y_2^2-y_3^2,\\
(h_1+h_2)\,x_2^3\,y_0-3\,(h_1-h_2)\,x_2^2\,x_3\,y_1-2\,x_2^2\,y_0\,y_1-\\
-3\,(h_1-h_2)\,x_2\,x_3^2\,y_0+(h_1-h_2)\,x_3^3\,y_1-2\,x_3^2\,y_0\,y_1\rangle\\
\mathscr{J}_6=\langle x_0,x_1,x_2,x_3\rangle.
\end{gathered}$$

- decomposition of the vanishing set of $\mathscr{I}$

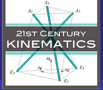
$$\mathscr{V}(\mathscr{I})=\bigcup_{i=1}^{5}\mathscr{V}(\mathscr{J}_i\cup\langle g_5,g_6,g_7,g_8\rangle)=\bigcup_{i=1}^{5}\mathscr{V}(\mathscr{K}_i)$$

Solving the system of equations

- solutions for generic parameters h_1, h_2 and d_1, d_2, d_3:

$$|\mathscr{V}(\mathscr{K}_1)| = |\mathscr{V}(\mathscr{K}_2)| = 2, \; |\mathscr{V}(\mathscr{K}_3)| = 4,$$

$$|\mathscr{V}(\mathscr{K}_4)| = 6, \; |\mathscr{V}(\mathscr{K}_5)| = 64.$$



Solving the system of equations

- solutions for generic parameters h_1, h_2 and d_1, d_2, d_3:

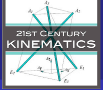
$$|\mathscr{V}(\mathscr{K}_1)| = |\mathscr{V}(\mathscr{K}_2)| = 2,\ |\mathscr{V}(\mathscr{K}_3)| = 4,$$

$$|\mathscr{V}(\mathscr{K}_4)| = 6,\ |\mathscr{V}(\mathscr{K}_5)| = 64.$$

- solutions for parameters with $d_1 = d_2 = d_3$:

$$|\mathscr{V}(\mathscr{K}_1)| = |\mathscr{V}(\mathscr{K}_2)| = |\mathscr{V}(\mathscr{K}_3)| = 2,$$

$$|\mathscr{V}(\mathscr{K}_4)| = 6,\ |\mathscr{V}(\mathscr{K}_5)| = 60$$



Solving the system of equations

- solutions for generic parameters h_1, h_2 and d_1, d_2, d_3:

$$|\mathscr{V}(\mathscr{K}_1)| = |\mathscr{V}(\mathscr{K}_2)| = 2, \; |\mathscr{V}(\mathscr{K}_3)| = 4,$$

$$|\mathscr{V}(\mathscr{K}_4)| = 6, \; |\mathscr{V}(\mathscr{K}_5)| = 64.$$

- solutions for parameters with $d_1 = d_2 = d_3$:

$$|\mathscr{V}(\mathscr{K}_1)| = |\mathscr{V}(\mathscr{K}_2)| = |\mathscr{V}(\mathscr{K}_3)| = 2,$$

$$|\mathscr{V}(\mathscr{K}_4)| = 6, \; |\mathscr{V}(\mathscr{K}_5)| = 60$$

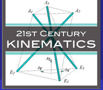
- “home pose” is solution of multiplicity 1 (SNU-3UPU $\rightarrow$ multiplicity 4)

Operation modes

- partial system $\mathscr{J}_i \cup \langle g_5, g_6, g_7 g_8 \rangle \longleftrightarrow$ operation mode
- five different modes:
  - translational mode, $\mathscr{J}_1 = \langle y_0, x_1, x_2, x_3 \rangle$
  - twisted translational mode, $\mathscr{J}_2 = \langle x_0, y_1, x_2, x_3 \rangle$
  - planar mode, $\mathscr{J}_3 = \langle y_0, y_1, x_2, x_3 \rangle$
  - upside-down planar mode, $\mathscr{J}_4 = \langle x_0, x_1, y_2, y_3 \rangle$
  - general mode, $\mathscr{J}_5 = \langle \dots \rangle$

Transformation matrix for translational mode

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2y_1 & 1 & 0 & 0 \\ -2y_2 & 0 & 1 & 0 \\ -2y_3 & 0 & 0 & 1 \end{pmatrix}$$



Manfred L. Husty, Hans-Peter Schröcker

Singular poses

- conditions on h_1, h_2, d_1, d_2, d_3 for singular poses are computable

Manfred L. Husty, Hans-Peter Schröcker

Singular poses

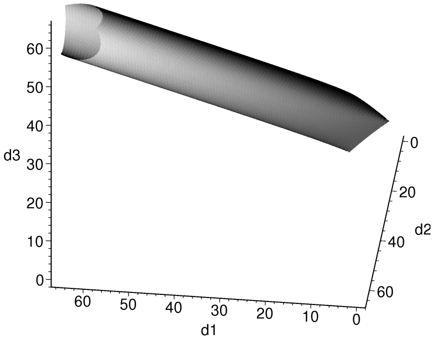
- conditions on h_1, h_2, d_1, d_2, d_3 for singular poses are computable
- Example: translational mode

$$d_1^4 + d_2^4 + d_3^4 - d_1^2\,d_2^2 - d_1^2\,d_3^2 - d_2^2\,d_3^2 -$$
$$-3\,(h_1 - h_2)^2\,(d_1^2 + d_2^2 + d_3^2) + 9\,(h_1 - h_2)^4 = 0$$

Singular poses

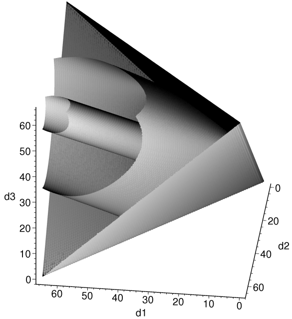
- conditions on h_1, h_2, d_1, d_2, d_3 for singular poses are computable
- Example: translational mode

$$d_1^4 + d_2^4 + d_3^4 - d_1^2\,d_2^2 - d_1^2\,d_3^2 - d_2^2\,d_3^2 -$$
$$-3\,(h_1 - h_2)^2\,(d_1^2 + d_2^2 + d_3^2) + 9\,(h_1 - h_2)^4 = 0$$



Singular poses

- Example: planar mode



$$F_1\,F_2\,(d_1+d_2-d_3)\,(d_1+d_3-d_2)\,(d_2+d_3-d_1)\,F_3=0$$

Changing operation modes

- change of operation mode only at special poses possible

Changing operation modes

- change of operation mode only at special poses possible
- dimensions of ideal intersections

	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$	$\mathcal{K}_4$	$\mathcal{K}_5$
$\mathcal{K}_1$	3	-1	2	-1	2
$\mathcal{K}_2$	-1	3	2	-1	2
$\mathcal{K}_3$	2	2	3	-1	2
$\mathcal{K}_4$	-1	-1	-1	3	2
$\mathcal{K}_5$	2	2	2	2	3

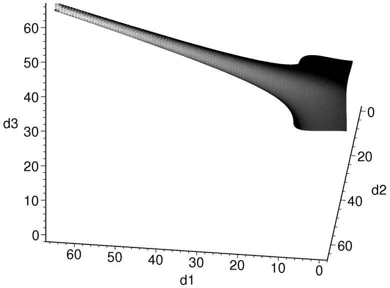
Changing operation modes

- mode change poses are also singular poses
- conditions on h_1, h_2, d_1, d_2, d_3 for such poses are computable

Changing operation modes

- mode change poses are also singular poses
- conditions on h_1, h_2, d_1, d_2, d_3 for such poses are computable
- Example: translational mode $\longleftrightarrow$ general mode

$$d_1^4 + d_2^4 + d_3^4 - d_1^2\, d_2^2 - d_1^2\, d_3^2 - d_2^2\, d_3^2 - 36\,(h_1 - h_2)^4 = 0$$

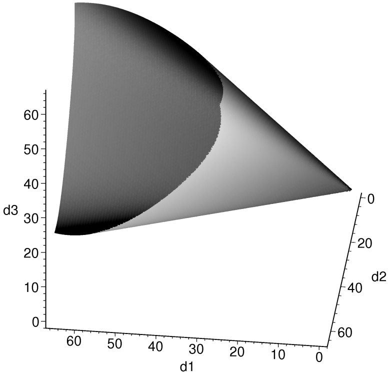


$h_1 = 12, h_2 = 7$

Changing operation modes

- Example: planar mode $\longleftrightarrow$ general mode

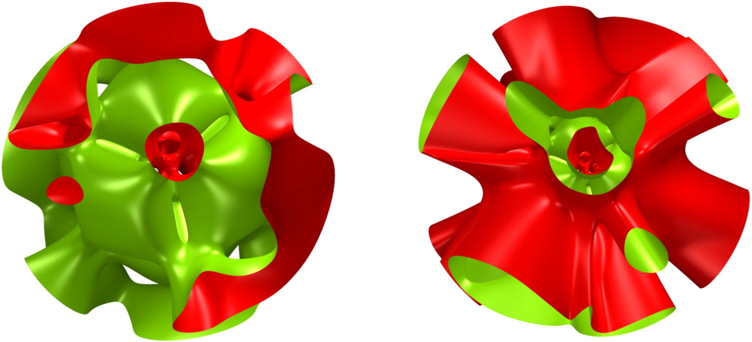
$$7\,(d_1^4+d_2^4+d_3^4)-11\,(d_1^2\,d_2^2-d_1^2\,d_3^2-d_2^2\,d_3^2)=0$$



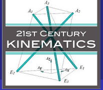
$h_1=12, h_2=7$

Most complicated transition are transitions to general mode

Most complicated transition are transitions to general mode



Transition surfaces of degree 24



Manfred L. Husty, Hans-Peter Schröcker

Synthesis of mechanisms

Changing the point of view the same constraint equations can be used for mechanism synthesis

Synthesis of mechanisms

Changing the point of view the same constraint equations can be used for mechanism synthesis

- Function synthesis
- Trajectory synthesis
- Motion synthesis

Synthesis of mechanisms

Changing the point of view the same constraint equations can be used for mechanism synthesis

- Function synthesis
- Trajectory synthesis
- Motion synthesis

Planar Burmester problem:

Given five poses of a planar system, construct a fourbar mechanism whose endeffector passes through all five poses
BURMESTER L. (19th century)

It is well known that the solution of this problem yields four dyads that can be combined to six four-bar mechanisms

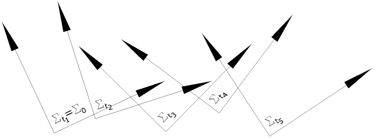


Figure: Five given poses

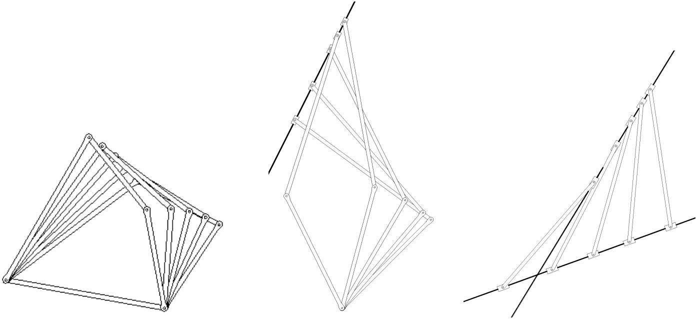


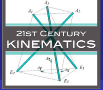
Figure: All possible four-bar mechanisms: a general one, a slider crank and a double slider mechanism

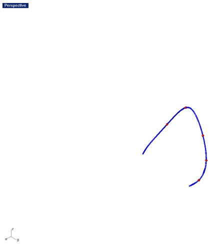
Here the expanded version of the constraint equation has to be used

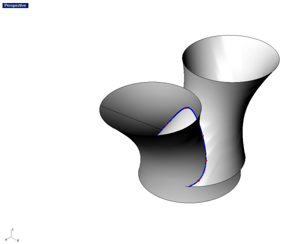
$$\begin{aligned}(R^2 - C_1^2 - C_2^2 - C_0(x^2+y^2) + 2C_1x + 2C_2y)X_0^2 \\ +(R^2 - C_1^2 - C_2^2 - C_0(x^2+y^2) - 2C_1x - 2C_2y)X_1^2 \\ +((4C_2x - 4C_1y)X_1 + (4C_0y - 4C_2)X_2 + (-4C_0x + 4C_1)X_3)X_0 \\ +((4C_1 + 4C_0x)X_2 + (4C_0y + 4C_2)X_3)X_1 - 4C_0X_3^2 - 4C_0X_2^2 = 0.\end{aligned} \tag{34}$$

X_i image space coordinates
C_i centers of the fixed pivots
x, y centers of the moving pivots


Perspective


21ST CENTURY
KINEMATICS


Perspective



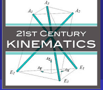
One of those points can be considered to be the point corresponding to the identity

$$(X_0 : X_1 : X_2 : X_3) = (1 : 0 : 0 : 0) \tag{35}$$

this simplifies the constraint equation

$$\begin{aligned}(-X_0X_3x + X_0X_2y + X_1X_2x + X_3X_1y - X_2^2 - X_3^2)C_0 - X_0X_2C_2 + X_0X_3C_1 \\ +X_0X_1xC_2 - X_1^2xC_1 + X_1X_2C_1 - X_0X_1yC_1 - X_1^2yC_2 + X_1X_3C_2 = 0\end{aligned} \tag{36}$$

Now the four remaining poses are given via their image space coordinates: $X_{ij}, j = 1 \ldots 4$.



Now the four remaining poses are given via their image space coordinates: $X_{ij}, j = 1 \ldots 4$.

It would be important for the designer to know in advance if among the synthesized mechanisms is a slider crank. This is the case if the following two conditions are fulfilled:

$$E1: \quad \left(-\frac{X_{13}(-X_{11}^2X_{02}X_{22}+X_{01}X_{21}X_{12}^2-X_{11}X_{31}X_{12}^2+X_{11}^2X_{12}X_{32})}{X_{11}X_{12}(X_{01}X_{12}-X_{11}X_{02})}+X_{23}\right)X_{03} - \frac{(X_{11}X_{31}X_{02}X_{12}-X_{01}X_{11}X_{12}X_{32}-X_{01}X_{21}X_{02}X_{12}+X_{01}X_{11}X_{02}X_{22})X_{13}^2}{X_{12}X_{11}(X_{01}X_{12}-X_{11}X_{02})} - X_{13}X_{33} = 0 \quad , \tag{37}$$

$$E2: \quad \left(-\frac{X_{14}(-X_{11}^2X_{02}X_{22}+X_{01}X_{21}X_{12}^2-X_{11}X_{31}X_{12}^2+X_{11}^2X_{12}X_{32})}{X_{11}X_{12}(X_{01}X_{12}-X_{11}X_{02})}+X_{24}\right)X_{04} - \frac{(X_{11}X_{31}X_{02}X_{12}-X_{01}X_{11}X_{12}X_{32}-X_{01}X_{21}X_{02}X_{12}+X_{01}X_{11}X_{02}X_{22})X_{14}^2}{X_{12}X_{11}(X_{01}X_{12}-X_{11}X_{02})} - X_{34}X_{14} = 0 \quad . \tag{38}$$

If a double slider is among the synthesized mechanisms then a third (more complicated compatability condition has to be fulfilled

Some examples

General four-bar mechanism

C_0	1	1
C_1	2	6
C_2	2	1
x	7,3821	9,1605
y	4,2434	1,1070

Table: Design parameter of mechanism 1

	pose 1	pose 2	pose 3	pose 4
a	-0,245005	-0,914683	-2,056744	-3,054058
b	0,523260	1,240571	2,235073	3,179009
ϕ	0.101061	0.116316	0.072202	-0.013746

Table: Given relative poses

Some examples

General four-bar mechanism

C_0	1	1
C_1	2	6
C_2	2	1
x	7,3821	9,1605
y	4,2434	1,1070

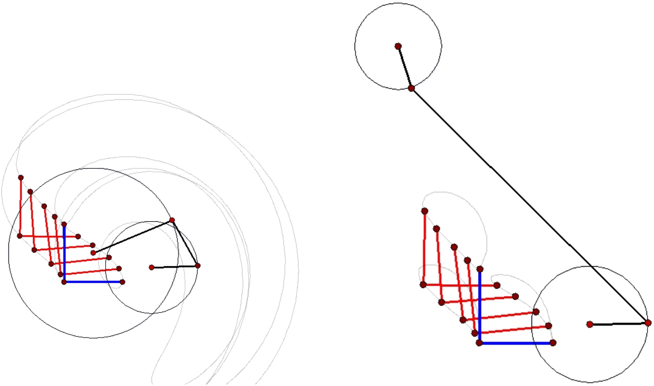
Table: Design parameter of mechanism 1

	pose 1	pose 2	pose 3	pose 4
a	-0,245005	-0,914683	-2,056744	-3,054058
b	0,523260	1,240571	2,235073	3,179009
ϕ	0.101061	0.116316	0.072202	-0.013746

Table: Given relative poses

	solution 1	solution 2	solution 3	solution 4
C_0	1	1	1	1
C_1	-34.640483	1.999996	6.000008	-4.402381
C_2	-29.947423	2.000000	0.999996	16.136008
x	18.091483	7.382096	9.160473	-3.697626
y	17.844191	4.243444	1.106973	13.877304
$\Rightarrow R$	71.166696	5.830956	3.162275	2.366097

Table: Obtained results



show animation

Example: spherical Burmester problem

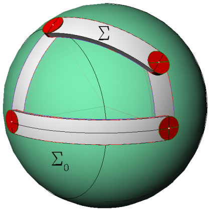
Given five poses of a spherical system, construct a four-bar mechanism whose endeffector passes through all five poses.

Example: spherical Burmester problem

Given five poses of a spherical system, construct a four-bar mechanism whose endeffector passes through all five poses.

Spherical circle constraint equation:

$$
\begin{aligned}
SCS: {}& 4Acx_0x_2 - 4Abx_0x_3 + 4Bax_0x_3 - 4Bcx_3x_2 - 4Cax_0x_2 - 4Cbx_3x_2 \\
& -2Aa - 2Bb - 2Cc + 4Bbx_3^2 + 4Ccx_2^2 + 4Aax_3^2 + 4Aax_2^2 + 4x_1^2Cc + \\
& 4x_1^2Bb - 4x_1Bcx_0 + 4x_1Cbx_0 - 4x_1Abx_2 - 4x_1Bax_2 - 4x_1Acx_3 \\
& -4x_1Cax_3 + B^2 + A^2 + C^2 + a^2 + b^2 + c^2 - r^2 = 0. \qquad (39)
\end{aligned}
$$



$$DCS: \quad \mathbf{w}^T \begin{pmatrix} \mathbf{I} & -2\mathbf{B} & \mathbf{0} \\ -2\mathbf{B} & \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & -1 \end{pmatrix} \mathbf{w} = 0 \tag{40}$$

Without loss of generality we can assume that the fixed system Σ_0 coincides with one of the five given orientations.

$$DCS_1 := -2Bb - 2Cc - 2Aa + A^2 + C^2 + B^2 + a^2 + b^2 + c^2 - r^2 = 0. \tag{41}$$

Now four simple equations are built by subtracting DCS_1 from the other four constraint equations:

$$M_{1j} = DCS_j - DCS_1, \qquad j = 2, \ldots 5.$$

The four difference equations are bilinear in the unknowns A, B, C, a, b, c and do not contain r.

Solution algorithm:

- Two of these equations, say $M_{1,2}$ and $M_{1,3}$ are used to solve linearly for two of the unknowns, say a, b.
- The solutions are substituted into $M_{1,4}$ and $M_{1,5}$. This yields two cubic equations C_1, C_2.
- The resultant of C_1, C_2 with respect to one of the remaining unknowns, say B yields a univariate polynomial Q^9 of degree nine in the unknown A.
- Q^9 factors into the solution polynomial Q^6 of degree six and in three linear factors.

Remarks:

- the univariate can be computed without specifying the pose parameters!
- Branch defect can also be easily detected with this approach!

Brunnthaler, Schrčker, and H., Synthesis of spherical four-bar mechanisms using spherical kinematic mapping. Advances in Robot Kinematics, 2006.

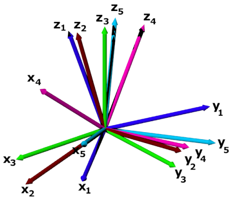
Schröcker and H., Kinematic mapping based assembly mode evaluation of spherical four-bar mechanisms. Proceedings of IFToMM 2007, Besanccon, 2007.

Example

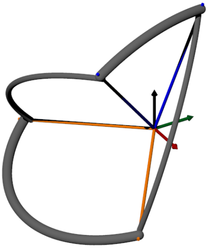
	x_0	x_1	x_2	x_3
Pose1	1	0	0	0
Pose2	0.37721	0.82336	0.38967	0.16722
Pose3	0.0078934	-0.041131	0.085164	-0.99549
Pose4	0.039457	0.77456	-0.60494	-0.18041
Pose5	-0.30301	-0.36492	0.85697	0.20157

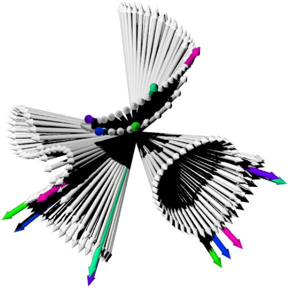
Table: Input data for the example

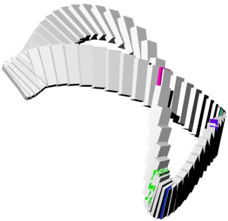
This example yields six real dyads that can be combined to 15 real spherical four-bars.



Five input poses


One solution four-bar


Motion of the coordinate frame


Motion of a rigid body