Kinestatic Analyses of Mechanisms with Compliant Elements

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How can such a simple mechanism have such a high order solution?

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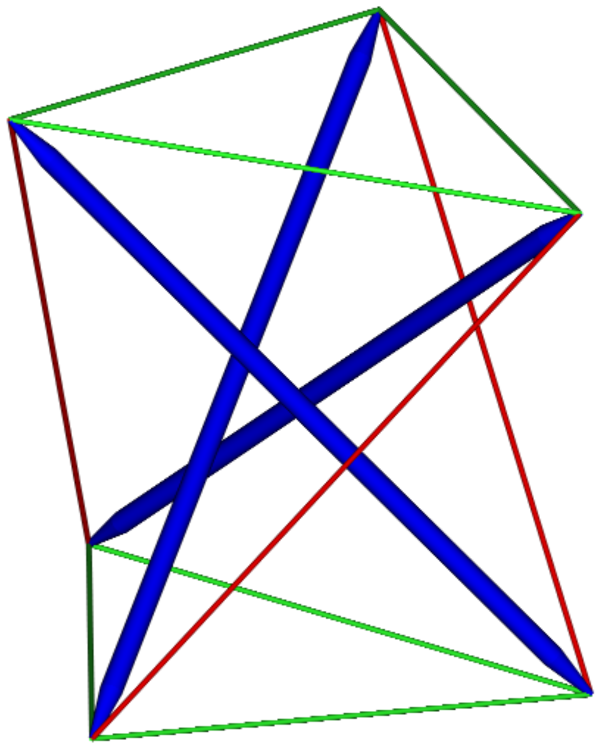
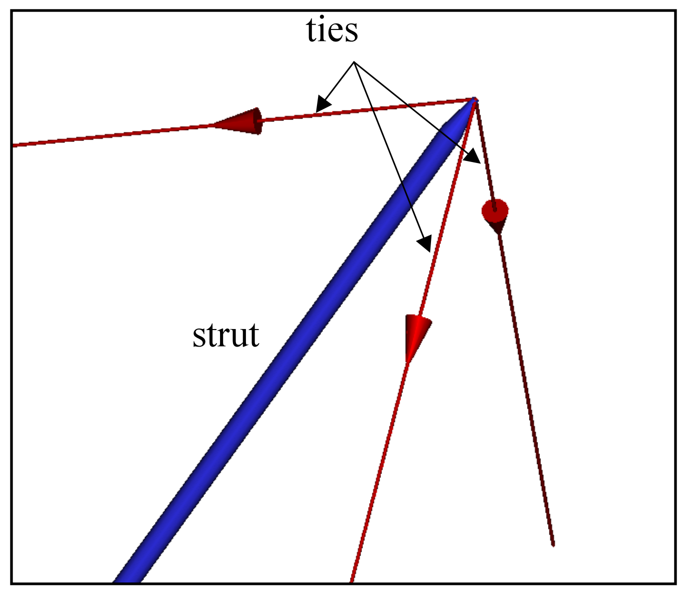
Tensegrity structures

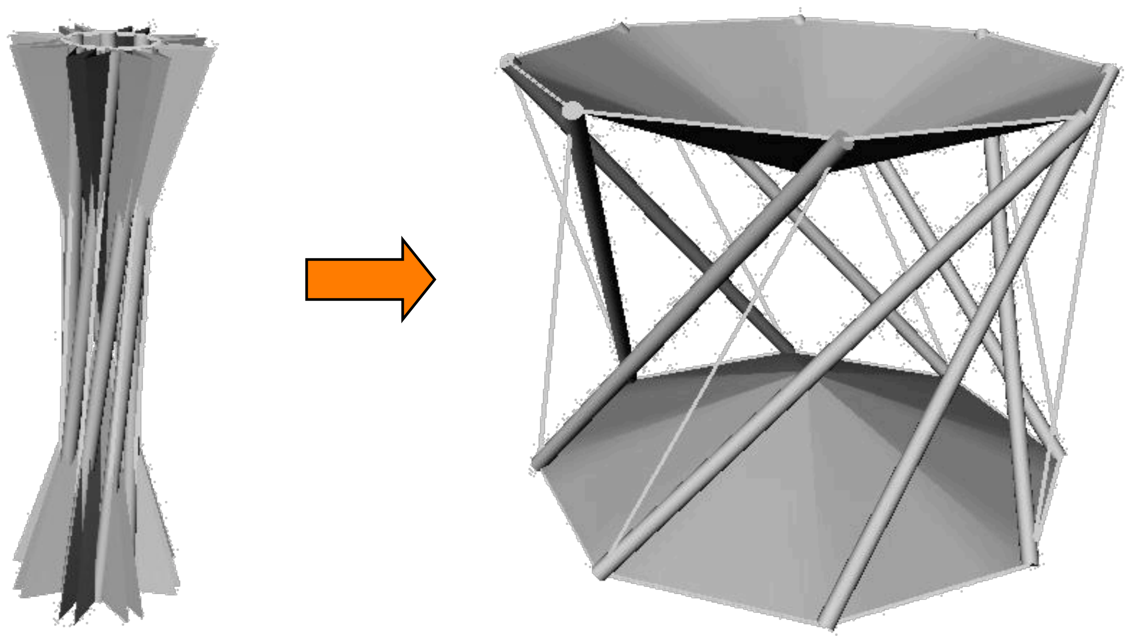
- comprised of struts in compression and ties in tension





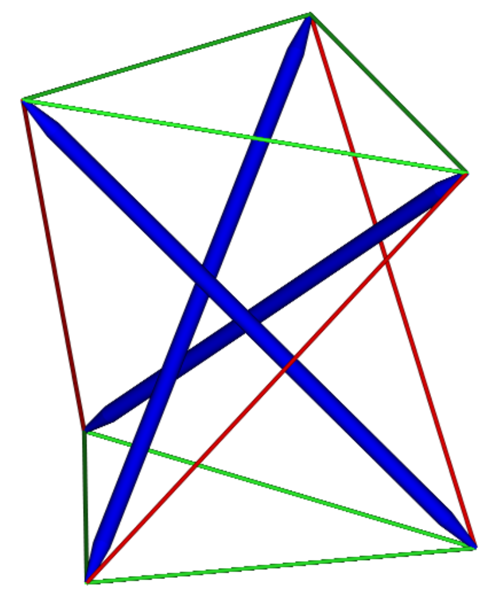
Self-deployable tensegrity structures

- certain ties replaced by elastic members



Can we solve the basic problem?

- determine in closed-form all equilibrium configurations of a self-deployable tensegrity structure given:
  - strut lengths
  - tie lengths
  - free lengths and spring constant of elastic members
  - any applied loads

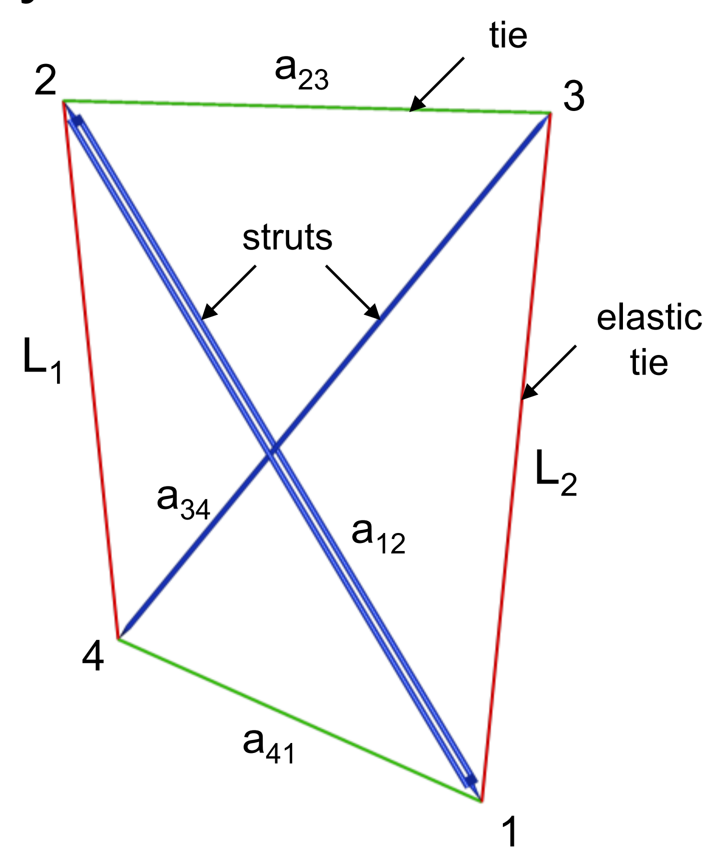


- Stern [1999] performed closed-form analysis of unloaded symmetric systems
  - 2 solutions, n=3..6
- Correa [2001] obtained numerical solution for general loaded systems
  - numerical convergence to a solution

Let's start with a warm up problem.

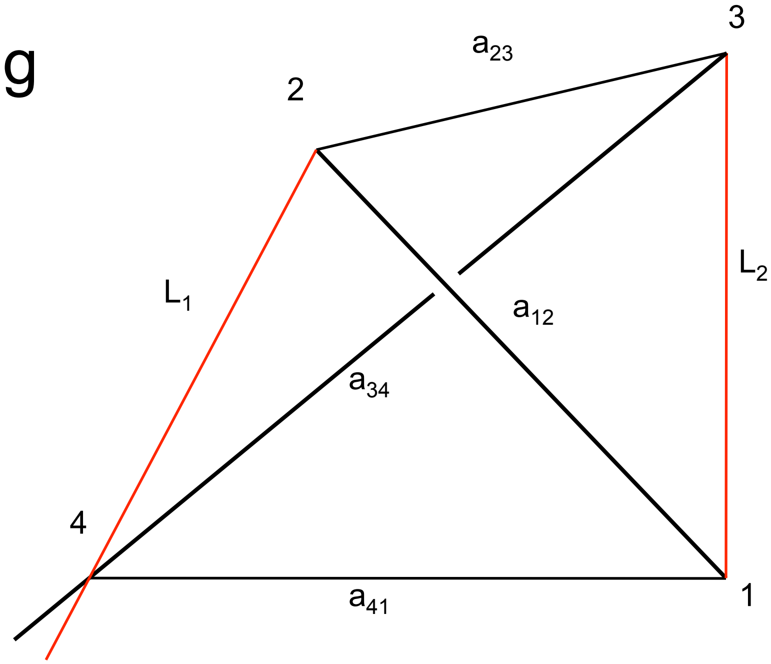
Planar 2-strut 2-spring tensegrity structure

- given:
  - strut lengths a_{12}, a_{34}
  - tie lengths a_{41}, a_{23}
  - spring parameters k_1, L_{01}, k_2, L_{02}
- determine:
  - all equilibrium poses



Planar 2-strut 2-spring tensegrity structure

- the struts and non-elastic ties form a simple 4-bar mechanism
- pose can be defined by one parameter
  - several descriptive parameters tried
- analysis was performed using an energy method and using L_1 as the descriptive parameter



Geometric constraints

$$A\,L_2^4 + B\,L_2^2 + C = 0$$

where

$$A = L_1^2\ ,\ B = L_1^4 + B_2\,L_1^2 + B_0,\ C = C_2\,L_1^2 + C_0$$

and where

B_2, B_0, C_2, and C_0 are expressed in terms of known quantities

Verification of geometric constraint equation

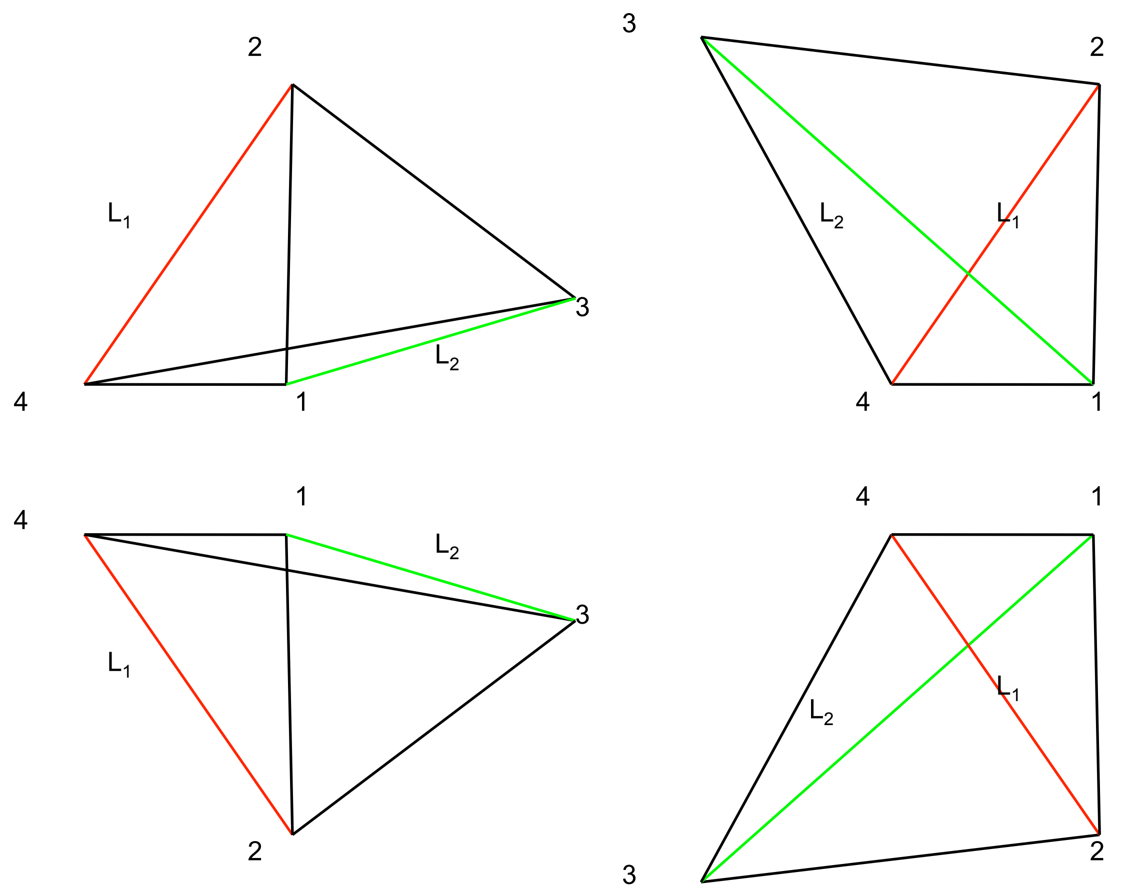


Figure 7: Possible Configurations for Numerical Example

Potential energy constraint

- at equilibrium, the potential energy in the springs will be a minimum

$$U = \tfrac{1}{2} k_1 (L_1 - L_{01})^2 + \tfrac{1}{2} k_2 (L_2 - L_{02})^2$$

- at a minimum potential energy state,

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_2 (L_2 - L_{02}) \frac{dL_2}{dL_1} = 0 \qquad (17)$$

- dL_2/dL_1 can be obtained via implicit differentiation of the geometry constraint (13) as

$$\frac{dL_2}{dL_1} = \frac{-L_1 [L_2^2 (L_2^2 + 2L_1^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{23}^2)(a_{41}^2 - a_{34}^2)]}{L_2 [L_1^2 (L_1^2 + 2L_2^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{41}^2)(a_{23}^2 - a_{34}^2))]} \qquad (18)$$

Geometry and Potential energy constraints

- geometry constraint

$$A L_2^4 + B L_2^2 + C = 0 \qquad (13)$$

- potential energy constraint

$$D L_2^5 + E L_2^4 + F L_2^3 + G L_2^2 + H L_2 + J = 0 \quad (19)$$

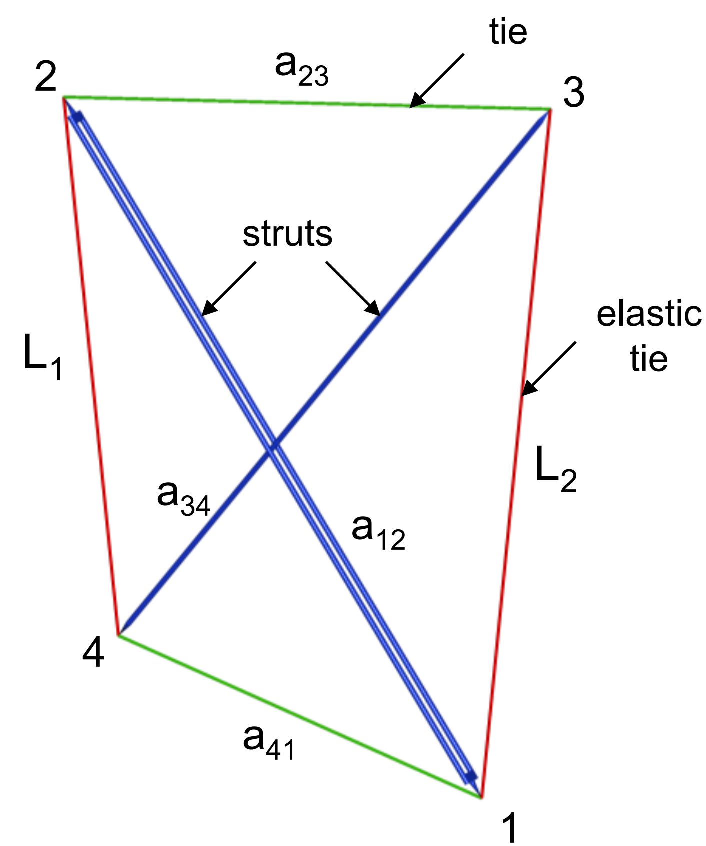
where the coefficients D through J are polynomials in L_1

- Sylvester’s elimination method can be used to identify the condition that the coefficients must satisfy in order for (13) and (19) to have common roots for L_2
  - multiply (13) by L_2, L_2^2, L_2^3, L_2^4
  - multiply (19) by L_2, L_2^2, L_2^3
  - obtain 9 ‘homogeneous’ equations in 9 unknowns

Sylvester's elimination

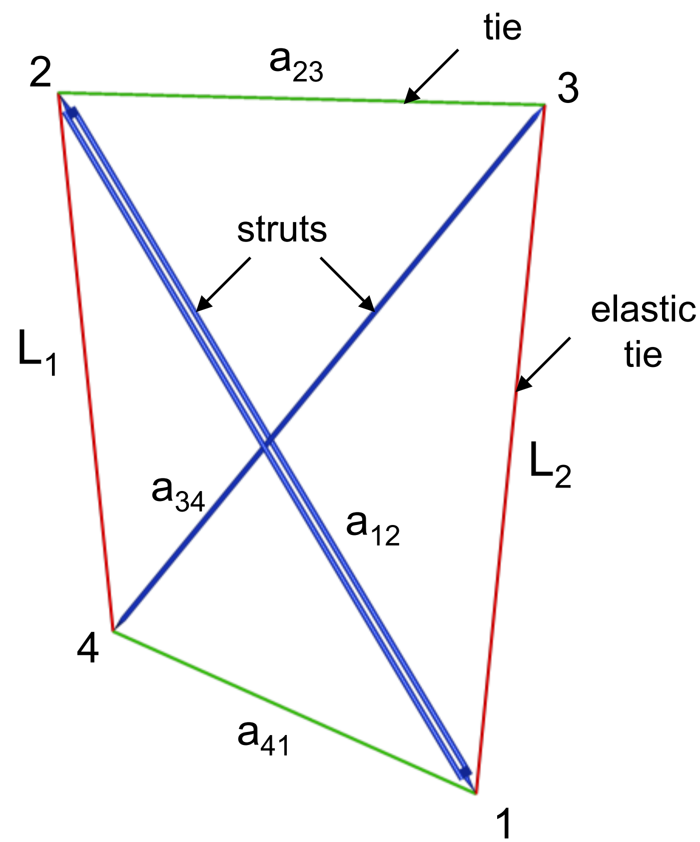
$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^{\,8} \\ L_2^{\,7} \\ L_2^{\,6} \\ L_2^{\,5} \\ L_2^{\,4} \\ L_2^{\,3} \\ L_2^{\,2} \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- determinant of coefficient matrix must equal zero which yields a 28th degree polynomial in L_1



Numerical example

- given:
  - a_{12} = 3 in. a_{34} = 3.5 in.
  - a_{41} = 4 in. a_{23} = 2 in.
  - L_{01} = 0.5 in. k_1 = 4 lbf/in.
  - L_{02} = 1 in. k_2 = 2.5 lbf/in.
- find L_1 and L_2 at equilibrium

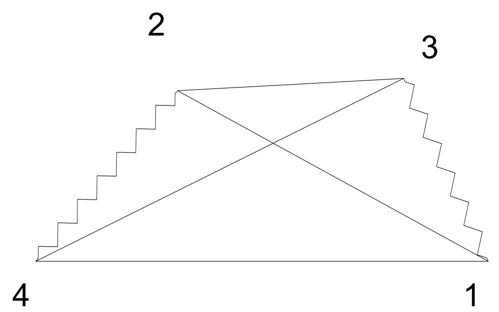


Numerical Example

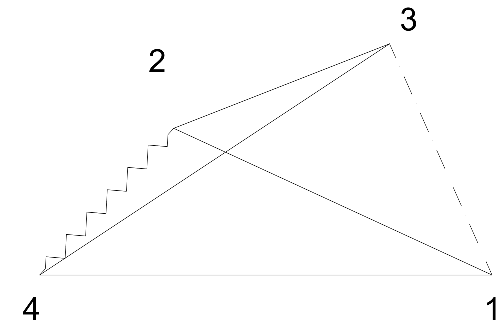
- results
  - coefficients of 28th degree polynomial in L_1 obtained
  - 8 real roots for L_1 with corresponding values for L_2
  - all 20 complex solution pairs (L_1, L_2) satisfied equations (13) and (19), i.e. geometry constraint and $dU/dL_1 = 0$
  - 4 cases correspond to minimum potential energy

Case	L_1, in.	L_2, in.
1	-5.4854	2.3333
2	-5.3222	-2.9009
3	-1.7406	-1.4952
4	-1.5760	1.8699
5	1.6280	1.7089
6	1.8628	-1.3544
7	5.1289	-3.2880
8	5.4759	2.3938

Numerical Example

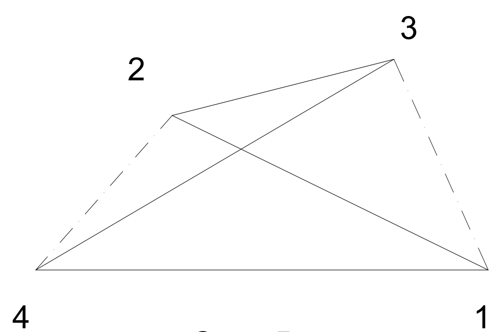


Case 3

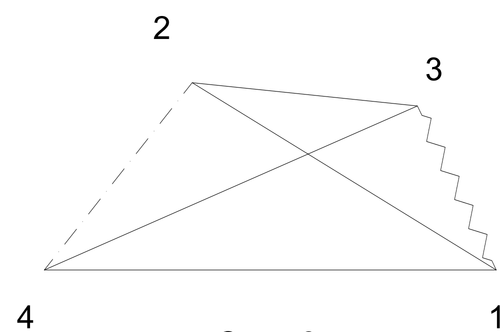


Case 4

spring in compression with a negative spring length



Case 5

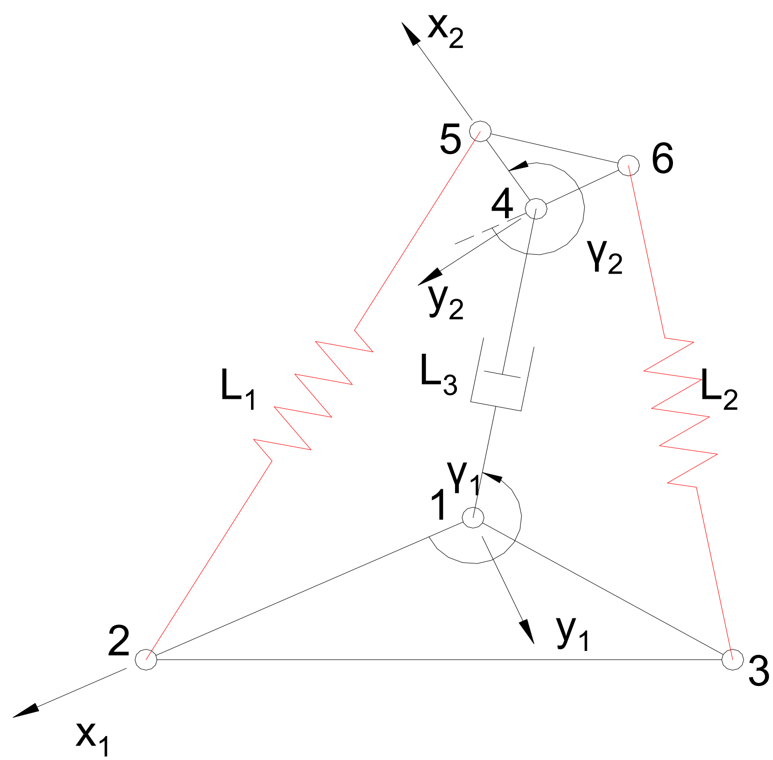


Case 6

spring in tension

Let's look at a second simple problem.

- In many papers involving compliant elements, researchers assume that their springs have a free length of zero.
- How much more complicated does the problem become if the spring free lengths are not zero?

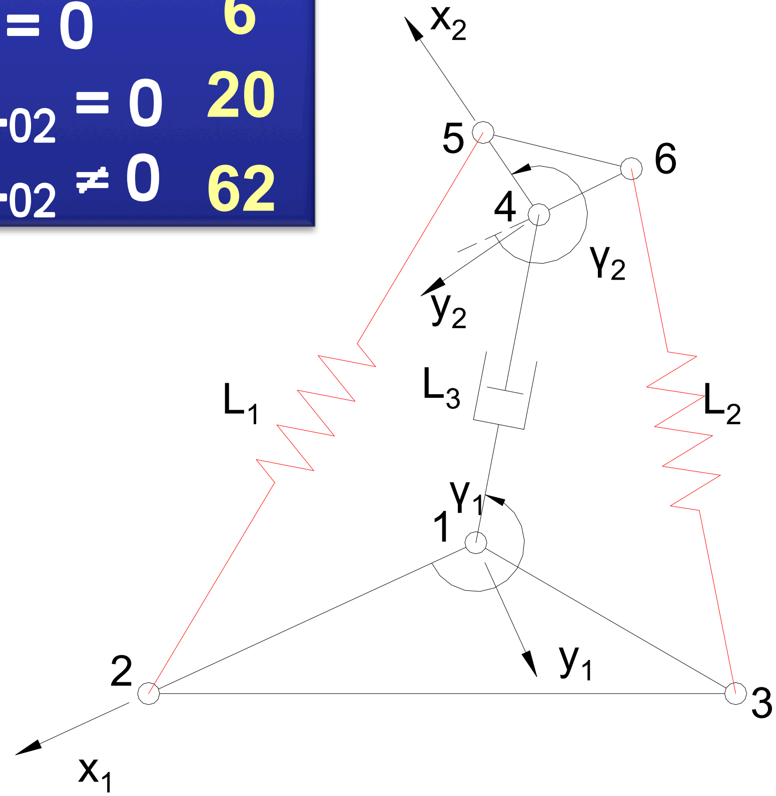


Problem Statement

- given:
  - L_{12}, p_{3x}, p_{3y}
  - L_{45}, p_{6x}, p_{6y}
  - L_3
  - k_1, L_{01}
  - k_2, L_{02}
- find:
  - γ_1 and γ_2 for all equilibrium configurations

3 cases

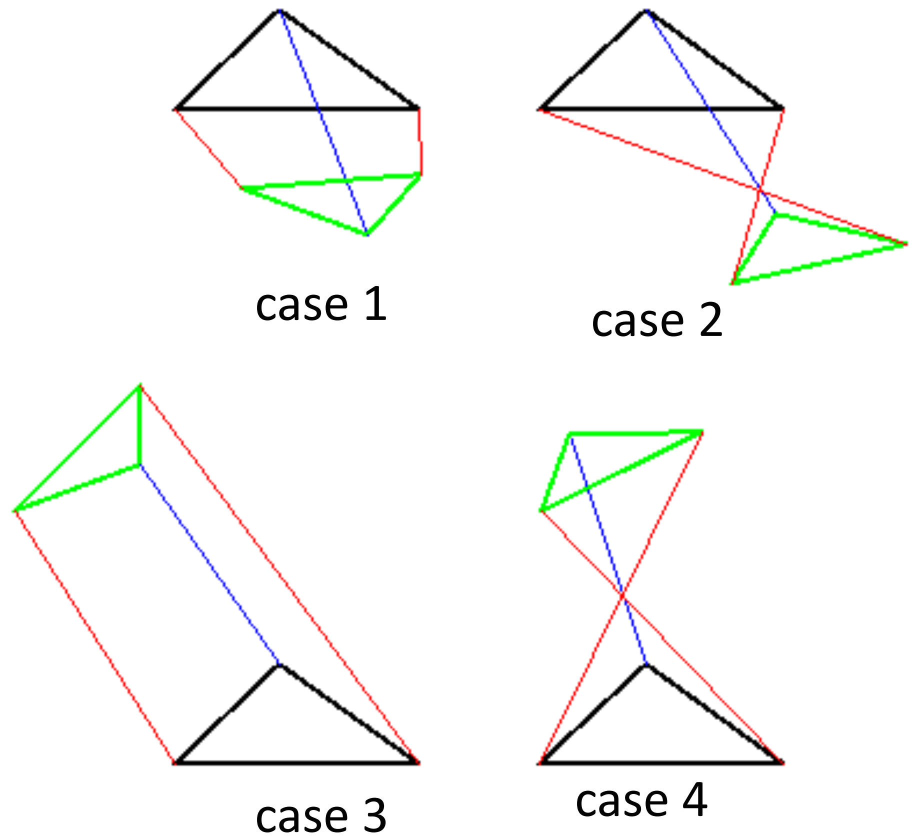
1. $L_{01} = L_{02} = 0$ **6**
2. $L_{01} \neq 0$, $L_{02} = 0$ **20**
3. $L_{01} \neq 0$, $L_{02} \neq 0$ **62**



Case 1: Numerical Example

L_{12} = 6 m,
p_{3x} = -1.25 m, p_{3y} = 6.887489 m,
L_{45} = 5.5 m,
p_{6x} = -1.14 m, p_{6y} = -3.13 m
L_3 = 10 m
k_1 = 2 N/m, L_{01} = 0,
k_2 = 3.5 N/m, L_{02} = 0

Solution #	γ_1, radians	γ_2, radians
1	1.1787	−1.1286
2	1.3704	2.1649
3	−1.7201	−0.4096
4	−2.0088	2.3924
5	-2.7032 + 1.1498 i	-2.6012 + 2.9712 i
6	-2.7032 – 1.1498 i	-2.6012 - 2.9712 i



Case 2: $L_{02} = 0$, $L_{01} \neq 0$

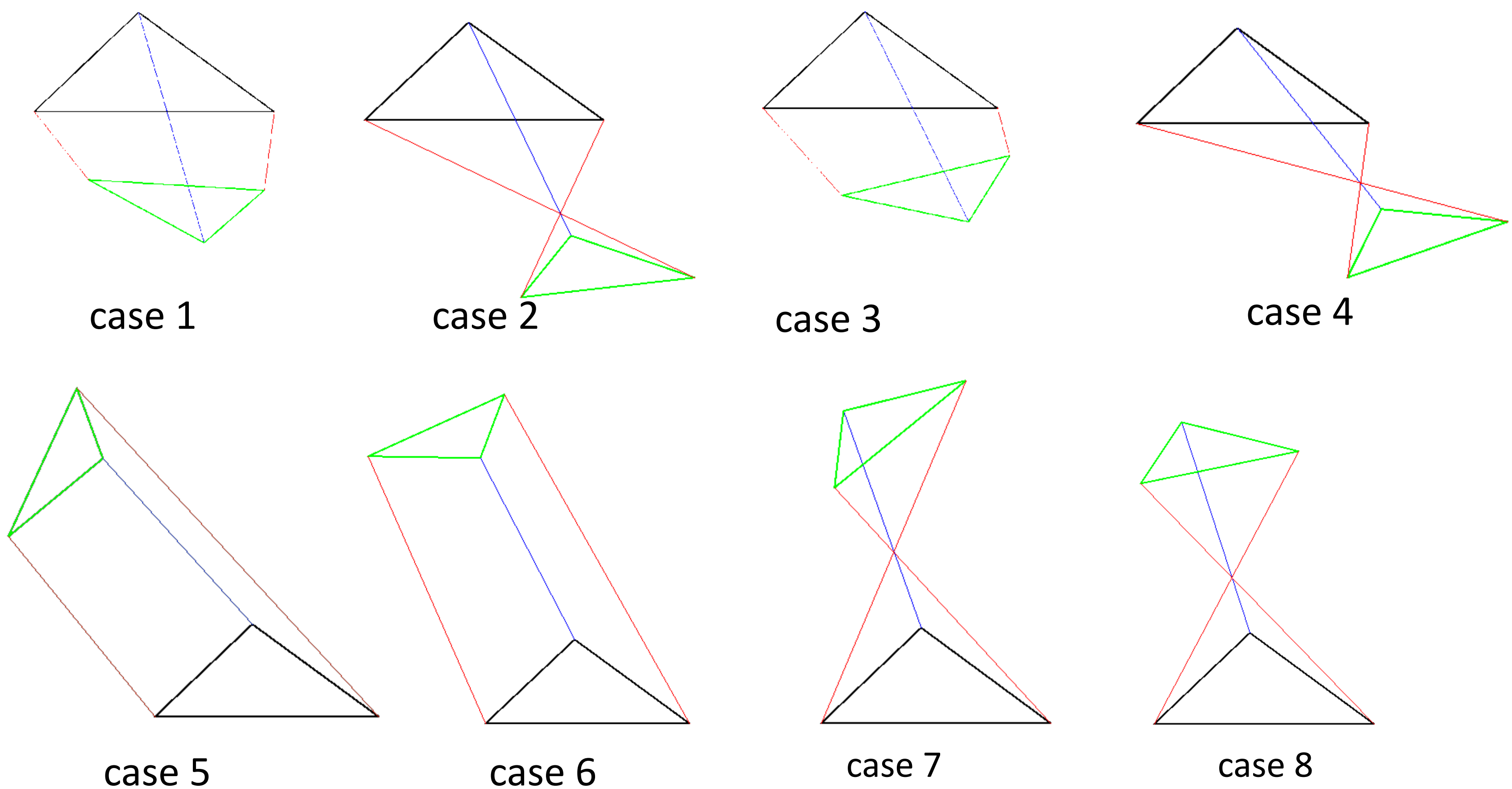
$$(E_1 x_2^2 + E_2 x_2 + E_3)\, d_1 + E_4 x_2^2 + E_5 x_2 + E_6 = 0$$
$$(F_1 x_2^2 + F_2 x_2 + F_3)\, d_1 + F_4 x_2^2 + F_5 x_2 + F_6 = 0$$
$$(G_1 x_2^2 + G_2 x_2 + G_3)\, d_1^2 + G_4 x_2^2 + G_5 x_2 + G_6 = 0$$

- Sylvester's Solution method
  - First two equations are multiplied by x_2, d_1, and d_1x_2, d_1^2 and $d_1^2x_2$. Third equation multiplied by x_2, d_1, and d_1x_2.
  - Results in 16 'homogeneous' equations in 16 unknowns.
  - The determinant of the coefficient matrix must equal zero.
  - Resulted in a 32nd degree polynomial in x_1.
  - Divided by $(1+x_1^2)^4$ and by square of the 2nd degree polynomial corresponding to $d_2 = 0$.

20th degree solution

Case 2: Numerical Example

- L_{01} = 2.3 m, 8 real solutions



Case 3: $L_{02} \neq 0$, $L_{01} \neq 0$

- Will obtain 4 equations of the form

$$(C_1 x_2^2 + C_2 x_2 + C_3) + (C_4 x_2^2 + C_5 x_2 + C_6)\, d_{2i} + (C_7 x_2^2 + C_8 x_2 + C_9)\, d_{1i} = 0 \quad (1)$$

$$(D_1 x_2^2 + D_2 x_2 + D_3) + (D_4 x_2^2 + D_5 x_2 + D_6)\, d_{2i} + (D_7 x_2^2 + D_8 x_2 + D_9)\, d_{1i} = 0 \quad (2)$$

$$(M_1 x_2^2 + M_2 x_2 + M_3) + (M_4 x_2^2 + M_5 x_2 + M_6)\, d_{1i}^2 = 0 \quad (3)$$

$$(N_1 x_2^2 + N_2 x_2 + N_3) + (N_4 x_2^2 + N_5 x_2 + N_6)\, d_{2i}^2 = 0 \quad (4)$$

where the coefficients are functions of x_1

Case 3: $L_{02} \neq 0$, $L_{01} \neq 0$

- Multiply Equations (1), (2) by
  - $\{1, d_{1i}, d_{2i}, d_{1i}^2, d_{2i}^2, d_{1i}d_{2i}, d_{1i}^2 d_{2i}, d_{1i}d_{2i}^2\} * \{1, x_2\}$
- Multiply Equation (3) by
  - $\{1, d_{1i}, d_{2i}, d_{1i}d_{2i}, d_{2i}^2\} * \{1, x_2\}$
- Multiply Equation (4) by
  - $\{1, d_{1i}, d_{2i}, d_{1i}d_{2i}, d_{1i}^2\} * \{1, x_2\}$
- Total Equations = 52
- Unknowns
  - $\{1, d_{1i}, d_{2i}, d_{1i}^2, d_{2i}^2, d_{1i}d_{2i}, d_{1i}^2 d_{2i}, d_{1i}d_{2i}^2, d_{1i}^3, d_{2i}^3, d_{1i}^3 d_{2i}, d_{1i}d_{2i}^3, d_{1i}d_{2i}^3\} * \{x_2^3, x_2^2, x_2, 1\}$

Case 3: $L_{02} \neq 0$, $L_{01} \neq 0$

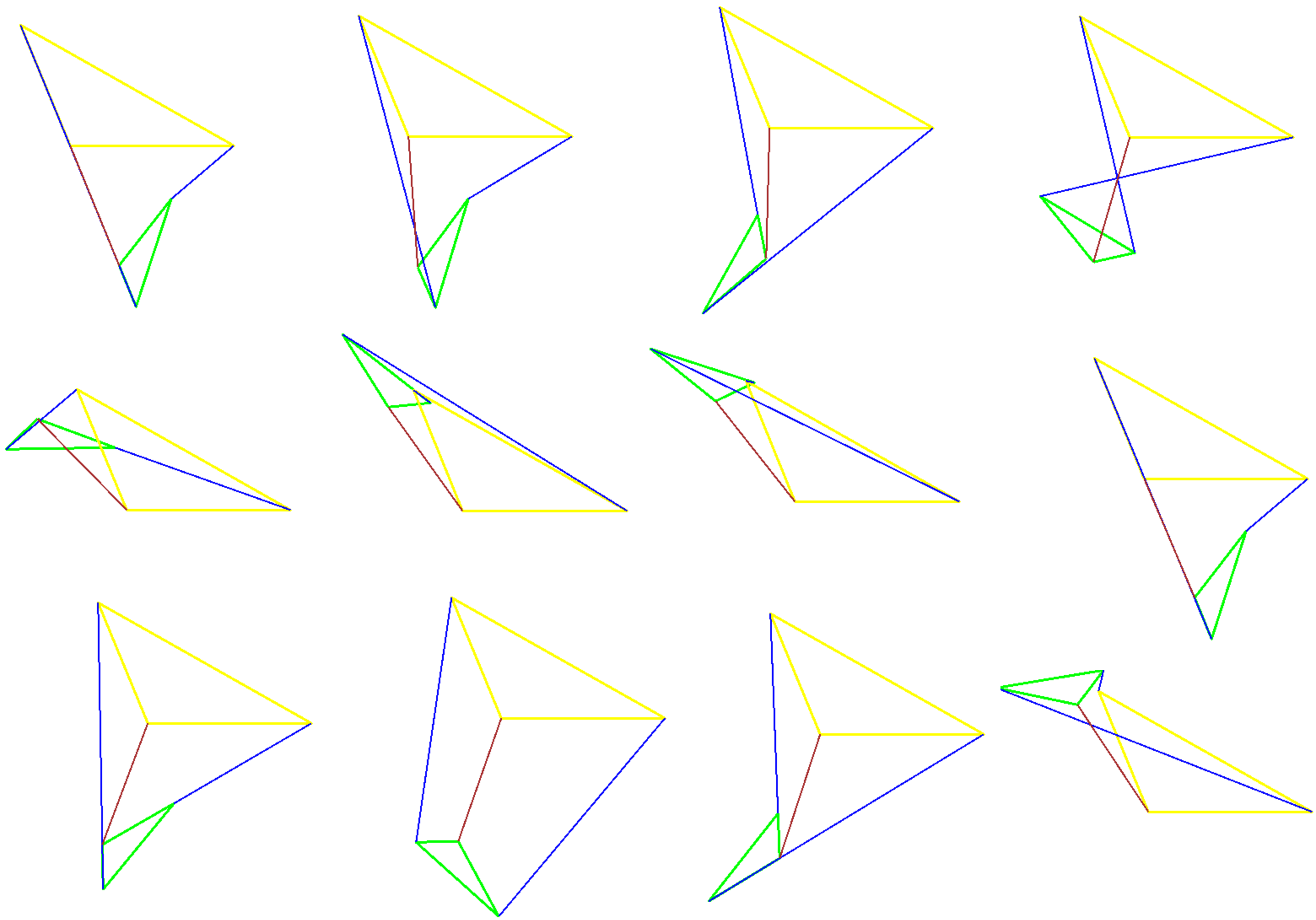
- Expansion of |**M**| = 0 yielded a 104th degree polynomial in x_1.
- This can be divided by $(1+x_1^2)^{13}$ to get 78th degree polynomial in x_1.
- Of the 78 solutions 16 were extraneous.
- 62 solutions were obtained.
- Numerical continuation method* found 88 solutions (62 + 26 circular points).

* *PHCpack software, Jan Verschelde, U. Illinois, Chicago*

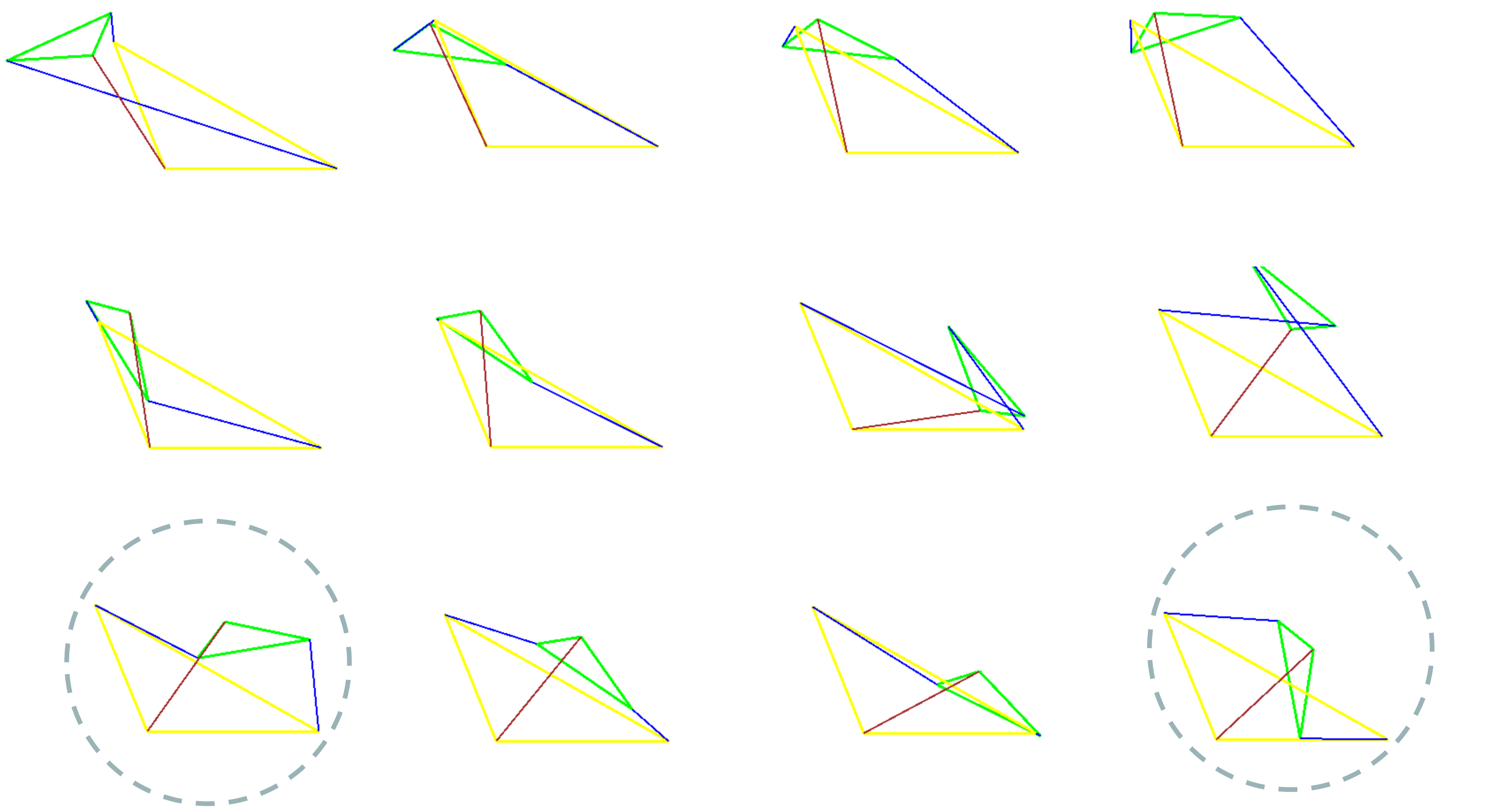
Case 3: $L_{02} \neq 0$, $L_{01} \neq 0$

- L_{01}= 5.1 m, L_{02} = 6.6309 m
- Out of the 62 solutions – 38 were complex and 24 were real
- All the solutions satisfy the four equations

Case 3: Real Solutions - I

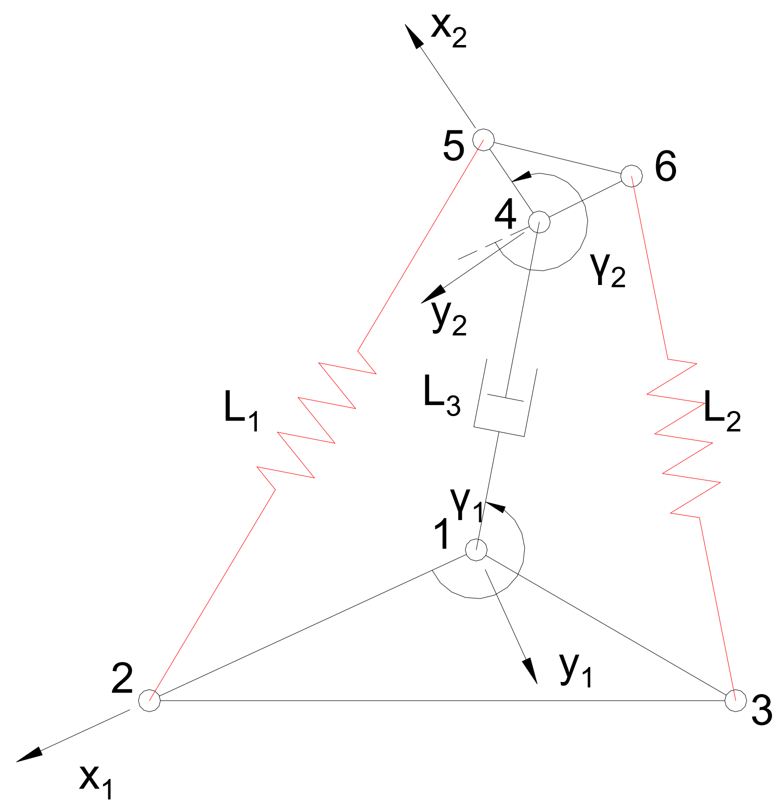


Case 3: Real Solutions - II



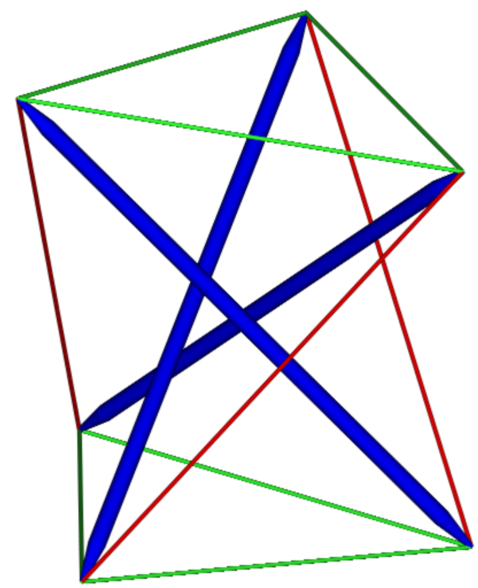
Conclusion

- Case 1, $L_{01} = L_{02} = 0$
  - 6 solutions
- Case 2: $L_{02} = 0, L_{01} \neq 0$
  - 20 solutions
- Case 3: $L_{02} \neq 0, L_{01} \neq 0$
  - 62 solutions

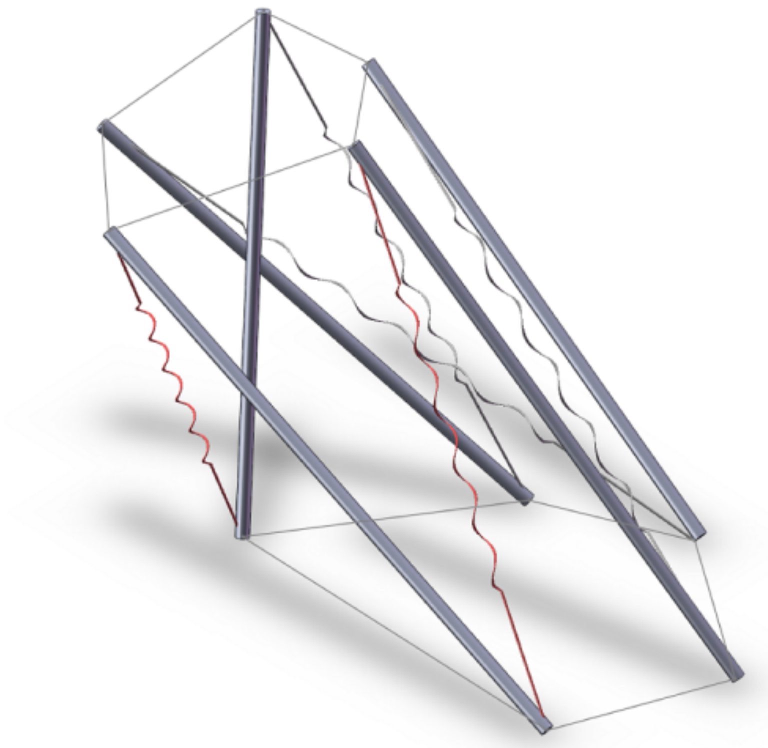


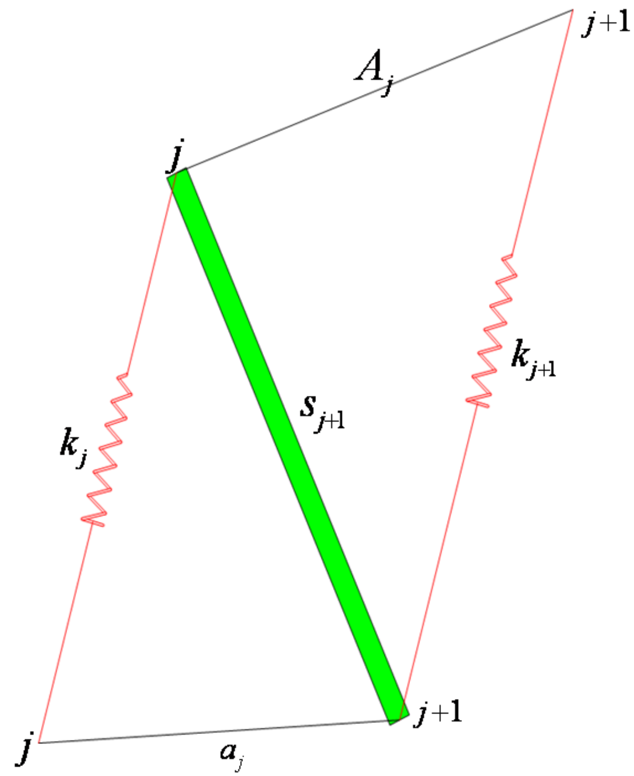
Back to the primary problem.

- given:
  - strut lengths
  - top tie lengths
  - bottom tie lengths
  - spring constant and free length of side ties
- find:
  - all equilibrium configurations



How to structure the problem?





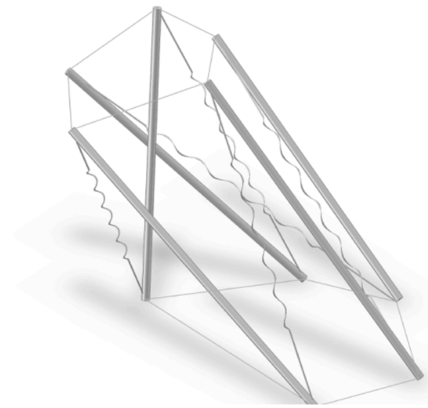
First, attempt to solve the problem when all the spring free lengths equal zero.

Problem Formulation

- given: A_j, a_j, k_j
- find: X_j, Y_j, Z_j, x_j, y_j, z_j

  such that

$$u = \frac{1}{2}\sum_{j=1}^{n} k_j \left(\left(X_j - x_j\right)^2 + \left(Y_j - y_j\right)^2 + \left(Z_j - z_j\right)^2 \right)$$

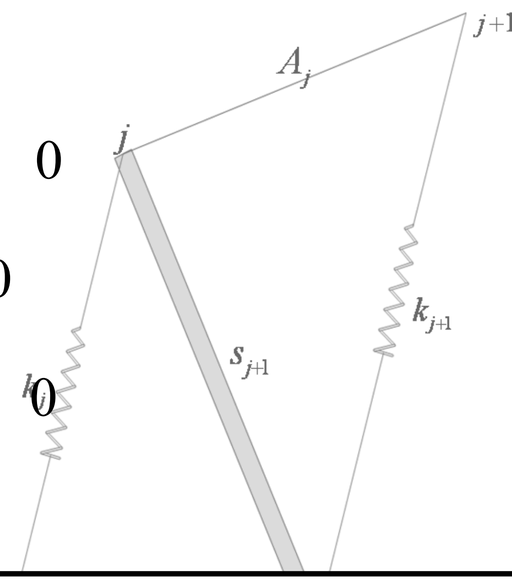


  is an extremum subject to

$$g_{1j} = \left(X_{j+1} - X_j\right)^2 + \left(Y_{j+1} - Y_j\right)^2 + \left(Z_{j+1} - Z_j\right)^2 - A_j^2 = 0$$

$$g_{2j} = \left(x_{j+1} - x_j\right)^2 + \left(y_{j+1} - y_j\right)^2 + \left(z_{j+1} - z_j\right)^2 - a_j^2 = 0$$

$$g_{3j} = \left(X_j - x_{j+1}\right)^2 + \left(Y_j - y_{j+1}\right)^2 + \left(Z_j - z_{j+1}\right)^2 - s^2 = 0$$



  for j = 1 ..n .

This formulation was selected to avoid any need to use tan half-angle substitutions to convert trigonometric functions into polynomials.

Example

Three strut tensegrity where spring free lengths equal zero.

There are 9 unknowns, i.e. X_j, Y_j, Z_j, j=1..3 .

There are 9 constraint equations:

$$kk_{11|2221}|\mathbf{\Delta\Delta}|^{22}|\mathbf{\Delta\Delta}| += \left| \quad \right|\left| \quad \right| \quad 0$$

$$kk_{22|3332}|\mathbf{\Delta\Delta}|^{22}|\mathbf{\Delta\Delta}| += \left| \quad \right|\left| \quad \right| \quad 0$$

$$kk_{33|1113}|\mathbf{\Delta\Delta}|^{22}|\mathbf{\Delta\Delta}| += \left| \quad \right|\left| \quad \right| \quad 0$$

$$(XX_{2121211}2{+}Y22A)^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$

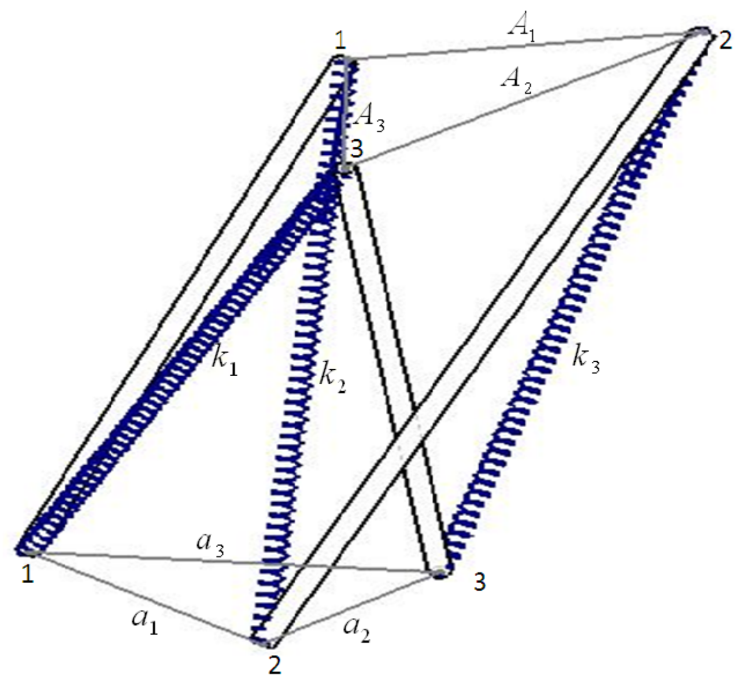
$$(Xx_{1212122}2{+}Z2{=})^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$

$$(XX_{3232322}2{+}Y{+}Z2{=})^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$

$$(Xx_{2323233}2{+}Z{+}2{=})^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$

$$(XX_{1313133}2{+}Y{+}Z2{=})^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$

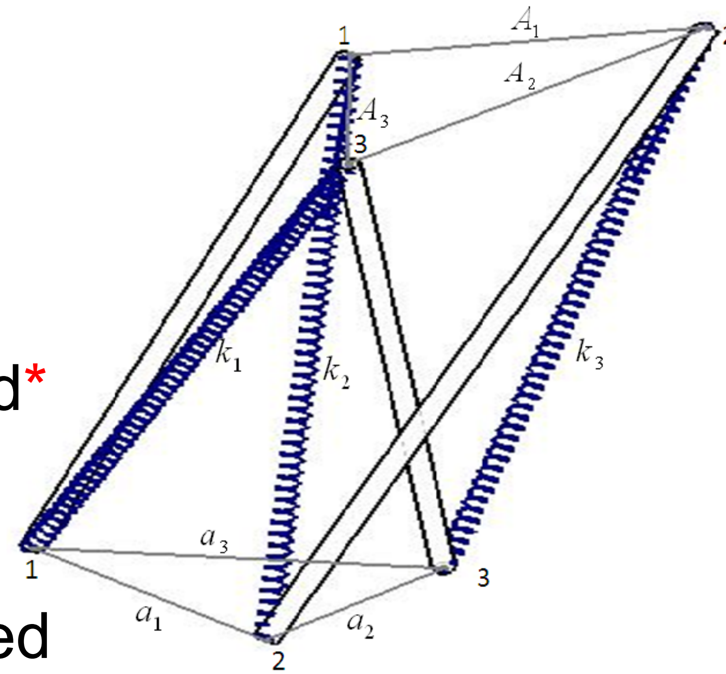
$$(Xx_{3131311}2{+}Z{+}2{=})^{222} \; (\quad) \; (\quad) \quad ^{2} \quad 0$$



Numerical Example

Three strut tensegrity

- a_1=10, a_2=12.3, a_3 = 15 cm
- s_1 = 20, s_2 = 23, s_3 = 10.5 cm
- k_1 = 3.8, k_2 = 3, k_3 = 4.3 N/cm



- The homotopy continuation method* was used to solve the set of 9 equations in 9 unknowns.
- 10 real configurations were obtained which satisfy the 9 equations (160 complex solutions obtained)

* PHCpack , http://www.math.uic.edu/~jan/download.html
Jan Verschelde, Univ. of Illinois at Chicago

Stability Analysis

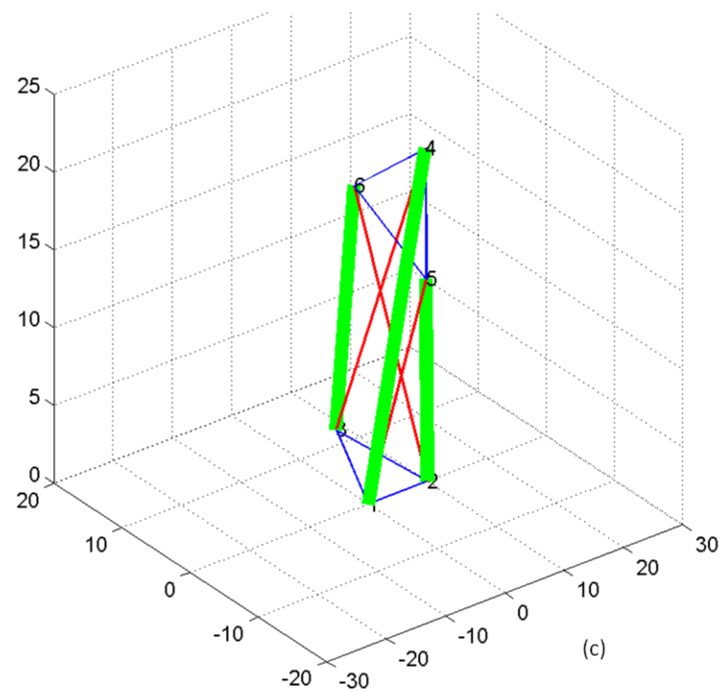
2^{nd} Order Analysis

- A 2^{nd} order analysis was conducted to classify the real solutions as
  - stable equilibrium
  - unstable equilibrium
  - neutral equilibrium
    - a small perturbation will continuously deform the structure to another neutral equilibrium state
    - have a statically balanced mechanism
- The equilibrium study becomes the examination of the positive definiteness of the Hessian matrix of the Lagrangian function *w*.

Numerical Example (cont)

stability analysis

- of the 10 real solutions
  - 1 is unstable (negative definite Hessian)
  - 7 are directionally stable (indefinite Hessian)
  - 2 are stable (positive definite Hessian)



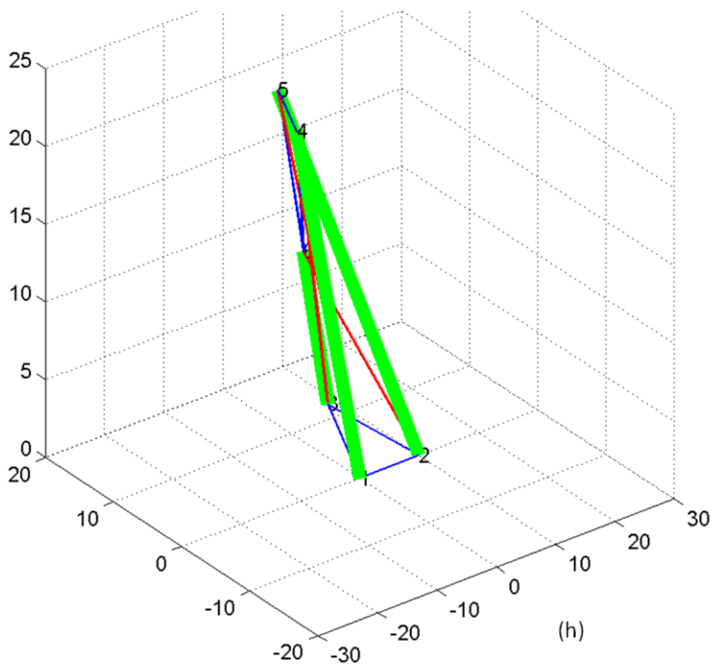
unstable case

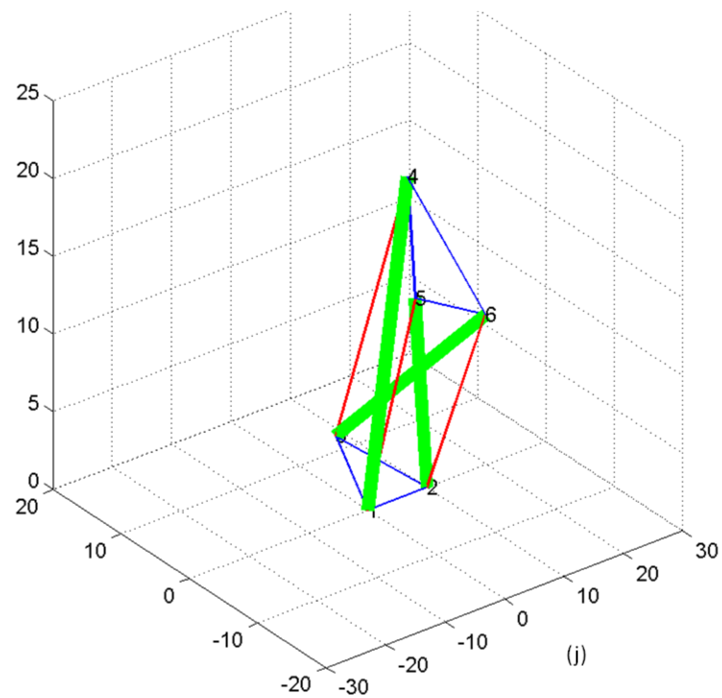
Numerical Example (cont)

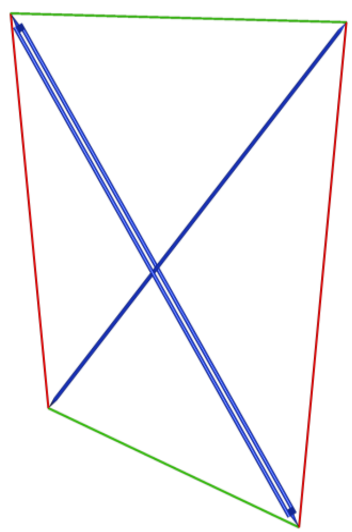
stability analysis

- of the 10 real solutions
  - 1 is unstable (negative definite Hessian)
  - 7 are directionally stable (indefinite Hessian)
  - 2 are stable (positive definite Hessian)

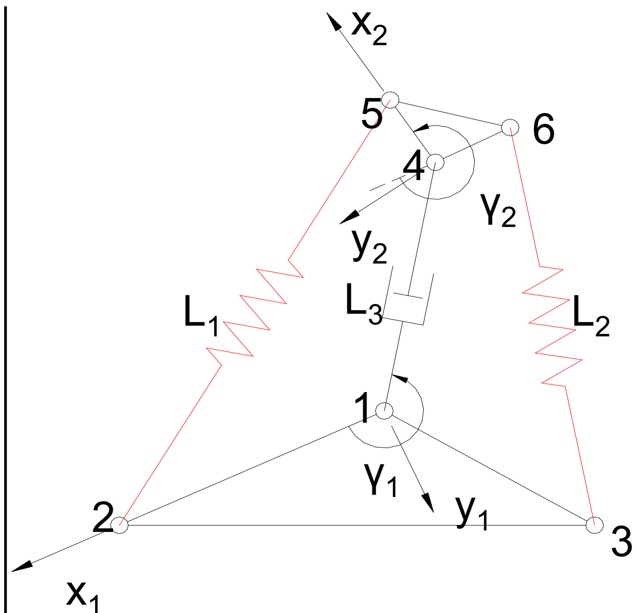
stable cases



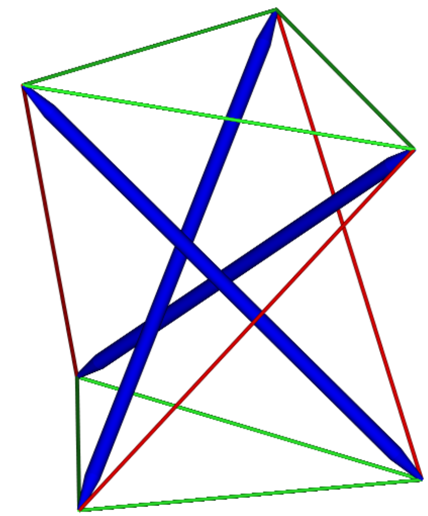




- 28^{th} degree univariate polynomial
- numerical example had 8 real roots, 4 of which were stable equilibrium
- 1 of the 4 stable equilibrium had both spring lengths > 0



- Case 1, $L_{01} = L_{02} = 0$
  - 6 solutions, 4 real
- Case 2: $L_{02} = 0$, $L_{01} \neq 0$
  - 20 solutions, 8 real
- Case 3: $L_{02} \neq 0$, $L_{01} \neq 0$
  - 62 solutions, 24 real
  - problem formulation has extraneous roots



- free lengths equal zero
- 9 equations in 9 unknowns
- continuation method yielded 10 real and 160 complex solutions
- 2 cases were in stable equilibrium