Four-bar Linkage Position Analysis

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Linkage in the World Frame

Four-bar Linkage

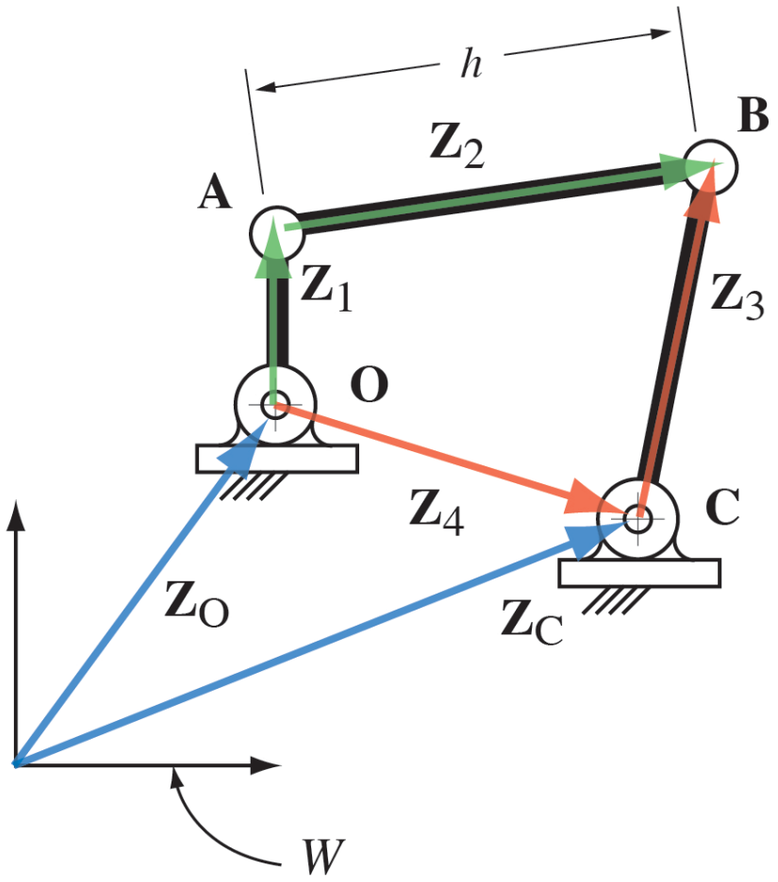
A four-bar linkage consists of two interconnected levers. The connecting link, or coupler, transfers the rotation of the input lever, or crank, to the output crank.

Locate the linkage in a reference position and let the coordinates of its fixed pivots be **O**, **C** and the coordinates of its moving pivots be **A**, **B**.

Introduce the vectors

Z_1=A-O, **Z_2=B-A**, **Z_3=B-C** and

Z_4=C-O, and **Z_O=O**, **Z_C=C**.



Closure Constraint

The constant length of the coupler constrains the rotation angles of the two cranks.

Rotation Operator

Rotation matrix

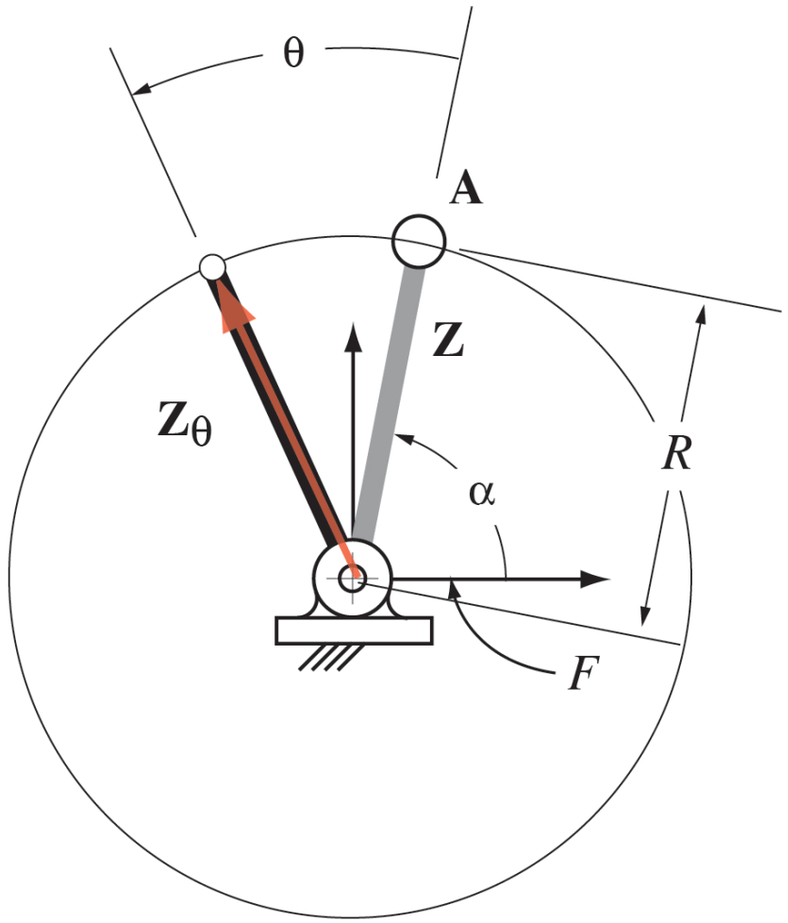
The planar rotation matrix rotates a vector **Z**=(x, y) by the angle θ:

$$\boldsymbol{\theta}\mathbf{Z} = \begin{Bmatrix} X(\theta) \\ Y(\theta) \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix},$$

The matrix $\boldsymbol{\theta}$ is an operator that rotates the vector **Z** from its initial position to its new position $\boldsymbol{\theta}\mathbf{Z}$.

To see this substitute $(x, y) = R(\cos\alpha, \sin\alpha)$ and compute,

$$\boldsymbol{\theta}\mathbf{Z} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} R\cos\alpha \\ R\sin\alpha \end{Bmatrix} = \begin{Bmatrix} R\cos(\theta+\alpha) \\ R\sin(\theta+\alpha) \end{Bmatrix}.$$



Linkage Closure Equation

The constant length of the coupler imposes a constraint on the input and output angles of the two cranks of the linkage.

The vector along the coupler link is given by

$$\boldsymbol{\varphi}\mathbf{Z}_2 = (\mathbf{Z}_C + \boldsymbol{\eta}\mathbf{Z}_3) - (\mathbf{Z}_O + \boldsymbol{\gamma}\mathbf{Z}_1),$$

or

$$\boldsymbol{\varphi}\mathbf{Z}_2 = \mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3 - \boldsymbol{\gamma}\mathbf{Z}_1.$$

This can also be written in the form,

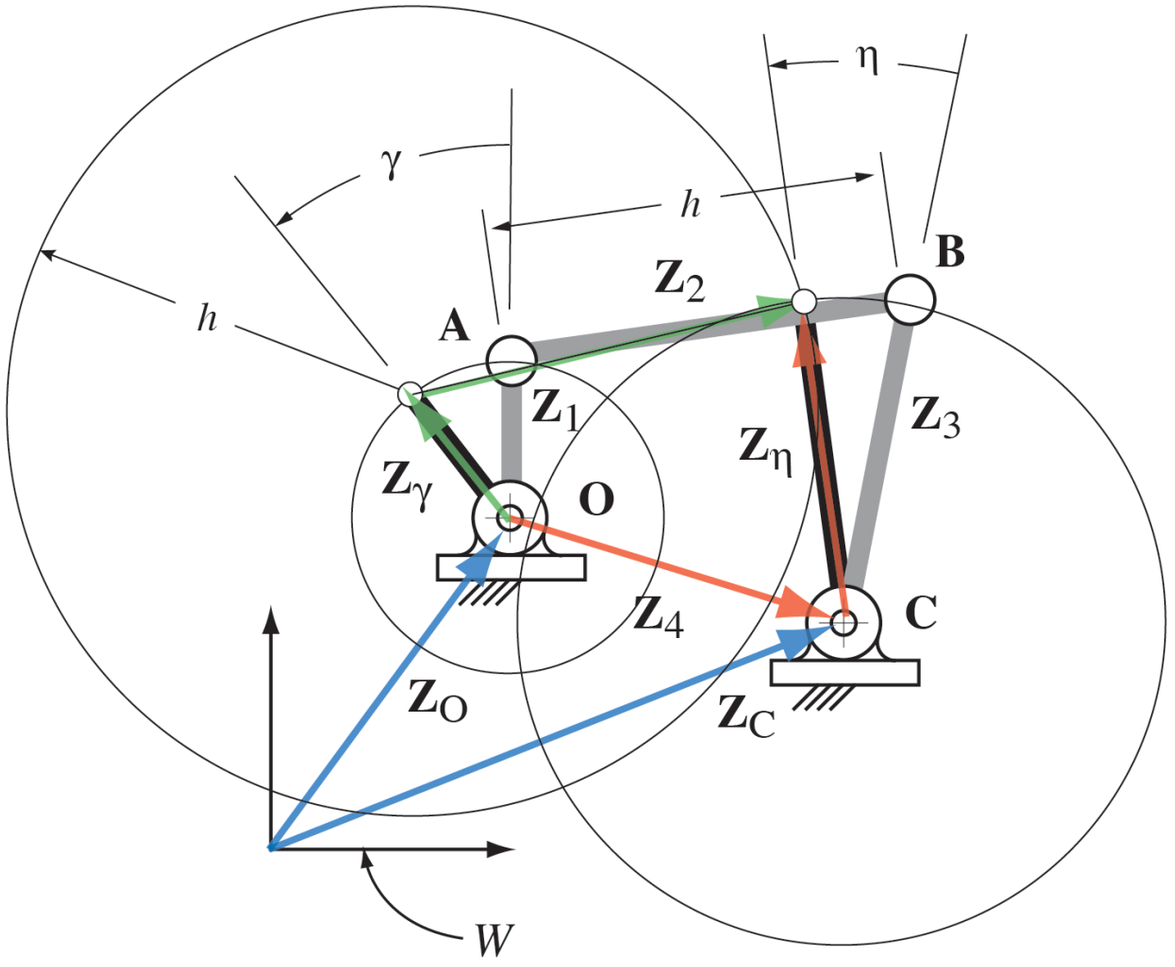
$$\boldsymbol{\gamma}\mathbf{Z}_1 + \boldsymbol{\varphi}\mathbf{Z}_2 = \mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3.$$

known as the *loop equation of the linkage*.

The length of the coupler link is given by

$$h^2 = (\mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3 - \boldsymbol{\gamma}\mathbf{Z}_1).(\mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3 - \boldsymbol{\gamma}\mathbf{Z}_1).$$

This is the *linkage closure equation*.



Expand the Linkage Closure Equation

Linkage Constraint Equation

Functions

Define the rotation operator

```
In[70]:= Ei[a_] := {{Cos[a], -Sin[a]}, {Sin[a], Cos[a]}}
```

Select coefficients of the constraint equation

```
In[71]:= Coeff[eqn_, a_] := Module[{Ac, Bc, Cc}, Ac = Coefficient[eqn, Cos[a]];
           Bc = Coefficient[eqn, Sin[a]]; Cc = Coefficient[eqn * Cos[a] * Sin[a], Cos[a] * Sin[a]];
           {Ac, Bc, Cc}]
```

Fixed an Moving Pivots

Fixed pivots in the reference position

```
In[72]:= Opt = {ox, oy}; Cpt = Opt + {g Cos[θ0], g Sin[θ0]};
```

Moving pivots in the reference position

```
In[73]:= Apt = Opt + {a Cos[γ0], a Sin[γ0]}; Bpt = Cpt + {b Cos[η0], b Sin[η0]};
```

Compute link vectors

```
In[74]:= Z1 = {a Cos[γ0], a Sin[γ0]};

In[75]:= Z3 = {b Cos[η0], b Sin[η0]};

In[76]:= Z4 = {g Cos[θ0], g Sin[θ0]};

In[77]:= Zγ = Ei[γ].Z1
Out[77]= {a Cos[γ] Cos[γ0] - a Sin[γ] Sin[γ0], a Cos[γ0] Sin[γ] + a Cos[γ] Sin[γ0]}

In[78]:= Zη = Ei[η].Z3
Out[78]= {b Cos[η] Cos[η0] - b Sin[η] Sin[η0], b Cos[η0] Sin[η] + b Cos[η] Sin[η0]}
```

Use Mathematica to expand the linkage closure equation.

Define the function Ei[θ_] to assemble the rotation matrix.

Then define the coordinates of the fixed and moving pivots in the reference position, Opt, Apt, But, and Cpt.

Use this data to define the link vectors, Z1, Z3, Z4, and compute Ei[γ]Z1, and Ei[η]Z3.

Expand the Linkage Closure Equation

Compute the Linkage Constraint Equation

```
In[79]:= Eqn = TrigExpand[(Z4 + Zη - Zγ).(Z4 + Zη - Zγ) - h^2]

Out[79]= a^2 + b^2 + g^2 - h^2 - 2 a b Cos[γ] Cos[γ0] Cos[η] Cos[η0] -
  2 a g Cos[γ] Cos[γ0] Cos[θ0] + 2 b g Cos[η] Cos[η0] Cos[θ0] +
  2 a b Cos[η] Cos[η0] Sin[γ] Sin[γ0] + 2 a g Cos[θ0] Sin[γ] Sin[γ0] -
  2 a b Cos[γ0] Cos[η0] Sin[γ] Sin[η] - 2 a b Cos[γ] Cos[η0] Sin[γ0] Sin[η] -
  2 a b Cos[γ0] Cos[η] Sin[γ] Sin[η0] - 2 a b Cos[γ] Cos[η] Sin[γ0] Sin[η0] +
  2 a b Cos[γ] Cos[γ0] Sin[η] Sin[η0] - 2 b g Cos[θ0] Sin[η] Sin[η0] -
  2 a b Sin[γ] Sin[γ0] Sin[η] Sin[η0] - 2 a g Cos[γ0] Sin[γ] Sin[θ0] -
  2 a g Cos[γ] Sin[γ0] Sin[θ0] + 2 b g Cos[η0] Sin[η] Sin[θ0] + 2 b g Cos[η] Sin[η0] Sin[θ0]
```

Collect coefficients for Acosη + Bsinη + C = 0.

```
In[80]:= {Ac, Bc, Cc} = Coeff[Eqn, η]

Out[80]= {-2 a b Cos[γ] Cos[γ0] Cos[η0] + 2 b g Cos[η0] Cos[θ0] + 2 a b Cos[η0] Sin[γ] Sin[γ0] -
   2 a b Cos[γ0] Sin[γ] Sin[η0] - 2 a b Cos[γ] Sin[γ0] Sin[η0] + 2 b g Sin[η0] Sin[θ0],
  -2 a b Cos[γ0] Cos[η0] Sin[γ] - 2 a b Cos[γ] Cos[η0] Sin[γ0] + 2 a b Cos[γ] Cos[γ0] Sin[η0] -
   2 b g Cos[θ0] Sin[η0] - 2 a b Sin[γ] Sin[γ0] Sin[η0] + 2 b g Cos[η0] Sin[θ0],
  a^2 + b^2 + g^2 - h^2 - 2 a g Cos[γ] Cos[γ0] Cos[θ0] + 2 a g Cos[θ0] Sin[γ] Sin[γ0] -
   2 a g Cos[γ0] Sin[γ] Sin[θ0] - 2 a g Cos[γ] Sin[γ0] Sin[θ0]}
```

Check for $\gamma = 0$ and $\eta = 0$

```
In[81]:= Eqn2 = Eqn /. {γ → 0, η → 0}

Out[81]= a^2 + b^2 + g^2 - h^2 - 2 a b Cos[γ0] Cos[η0] - 2 a g Cos[γ0] Cos[θ0] + 2 b g Cos[η0] Cos[θ0] -
  2 a b Sin[γ0] Sin[η0] - 2 a g Sin[γ0] Sin[θ0] + 2 b g Sin[η0] Sin[θ0]

In[82]:= Eqn3 = TrigExpand[(Bpt - Apt).(Bpt - Apt) - h^2]

Out[82]= a^2 + b^2 + g^2 - h^2 - 2 a b Cos[γ0] Cos[η0] - 2 a g Cos[γ0] Cos[θ0] + 2 b g Cos[η0] Cos[θ0] -
  2 a b Sin[γ0] Sin[η0] - 2 a g Sin[γ0] Sin[θ0] + 2 b g Sin[η0] Sin[θ0]
```

Use TrigExpand to compute the linkage closure equation.

Collect the coefficients of Cos[η] to define Ac, the coefficients of Sin[η] to define Bc, and all the remaining terms to define Cc.

The result is an equation of the form

$$A(\gamma)\cos\eta + B(\gamma)\sin\eta + C(\gamma) = 0.$$

For a given four-bar linkage we have the coordinates **O**, **A**, **B**, and **C**, so for a given value of the input angle γ, this equation determines the output η.

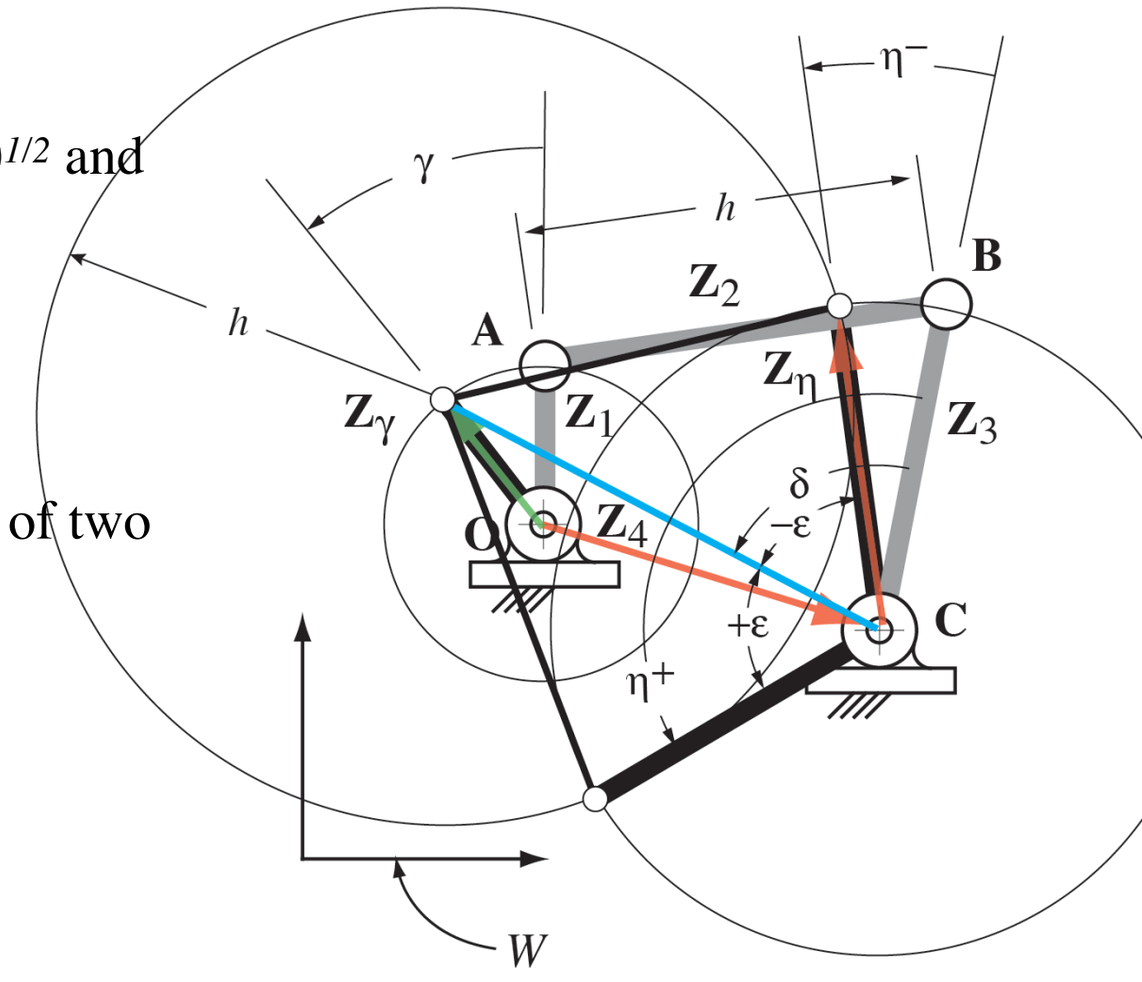
Solving the Closure Equation

For a given value γ, the linkage closure equation

$$A(\gamma)\cos\eta + B(\gamma)\sin\eta + C(\gamma) = 0,$$

is solved to determine the angle η, as follows.

- Introduce the angle δ, defined such that $\cos\delta = A/(A^2+B^2)^{1/2}$ and $\sin\delta = B/(A^2+B^2)^{1/2}$, so
  $\delta = \arctan(B/A)$;
- then the constraint equation becomes
  $\cos\delta\cos\eta + \sin\delta\sin\eta + C/(A^2+B^2)^{1/2} = 0;$
- The trigonometric identity for the cosine of the difference of two angles yields
  $\cos(\eta - \delta) = -C/(A^2+B^2)^{1/2}$,
- introduce $\varepsilon = \arccos(-C/(A^2+B^2)^{1/2})$, then we have the two solutions,
  $\eta^- = \delta - \varepsilon$ and $\eta^+ = \delta + \varepsilon$



Notice, the arccos() can be evaluated *only* if $-1 \le C/(A^2+B^2)^{1/2} \le 1$,
which is the condition on values for γ that allow the linkage to be assembled.

Compute the Coupler Angle

Once the input angle γ and output angle η are known, we determine the coupler angle φ by the following formula,

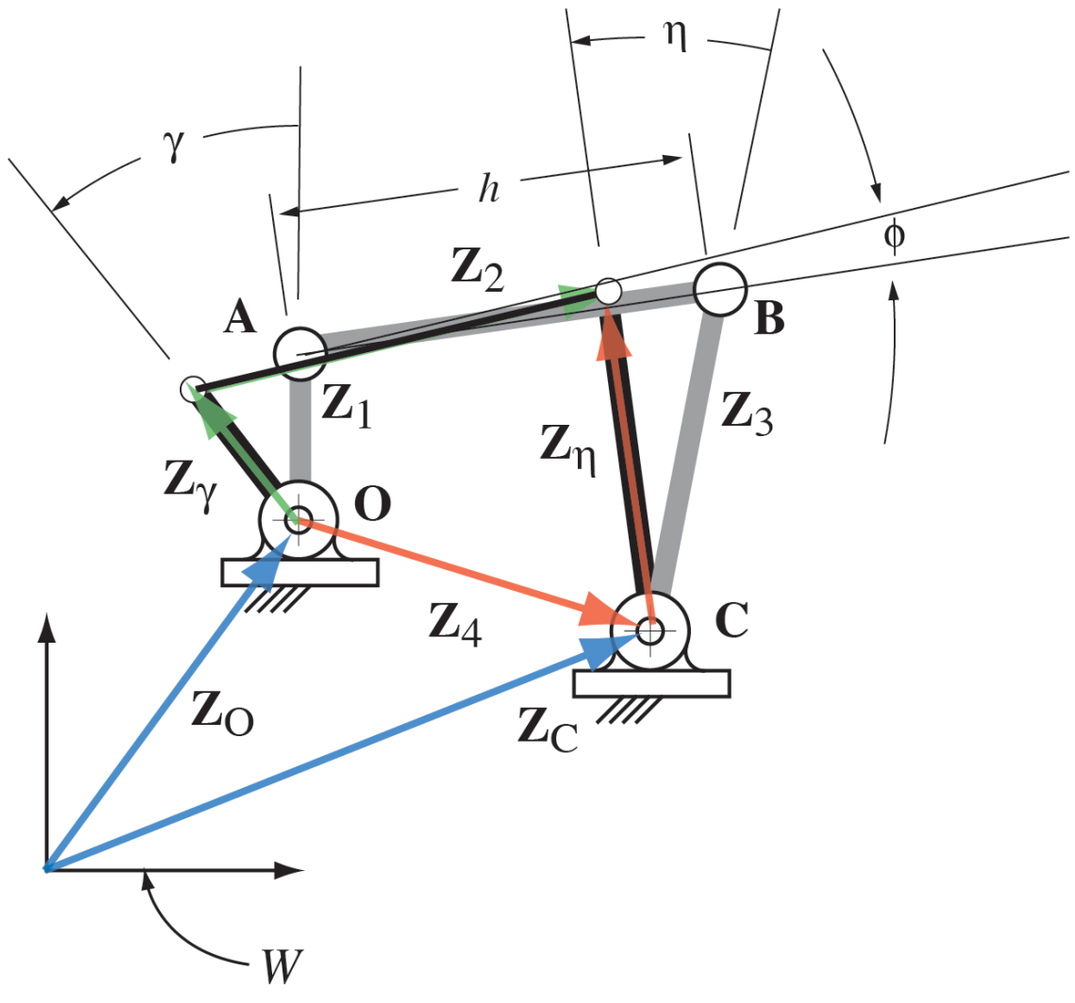
$$\cos\varphi = (\mathbf{B}\text{-}\mathbf{A}).(\mathbf{Z_4} + \boldsymbol{\eta}\mathbf{Z_3} - \boldsymbol{\gamma}\mathbf{Z_1}) = x_h,$$
$$\sin\varphi = (\mathbf{B}\text{-}\mathbf{A})\times(\mathbf{Z_4} + \boldsymbol{\eta}\mathbf{Z_3} - \boldsymbol{\gamma}\mathbf{Z_1}) = y_h.$$

The result is

$$\varphi = \arctan(x_h, y_h).$$

The cross product for two dimensional vectors is defined as

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} v_x & v_y \\ v_x & v_y \end{bmatrix}.$$

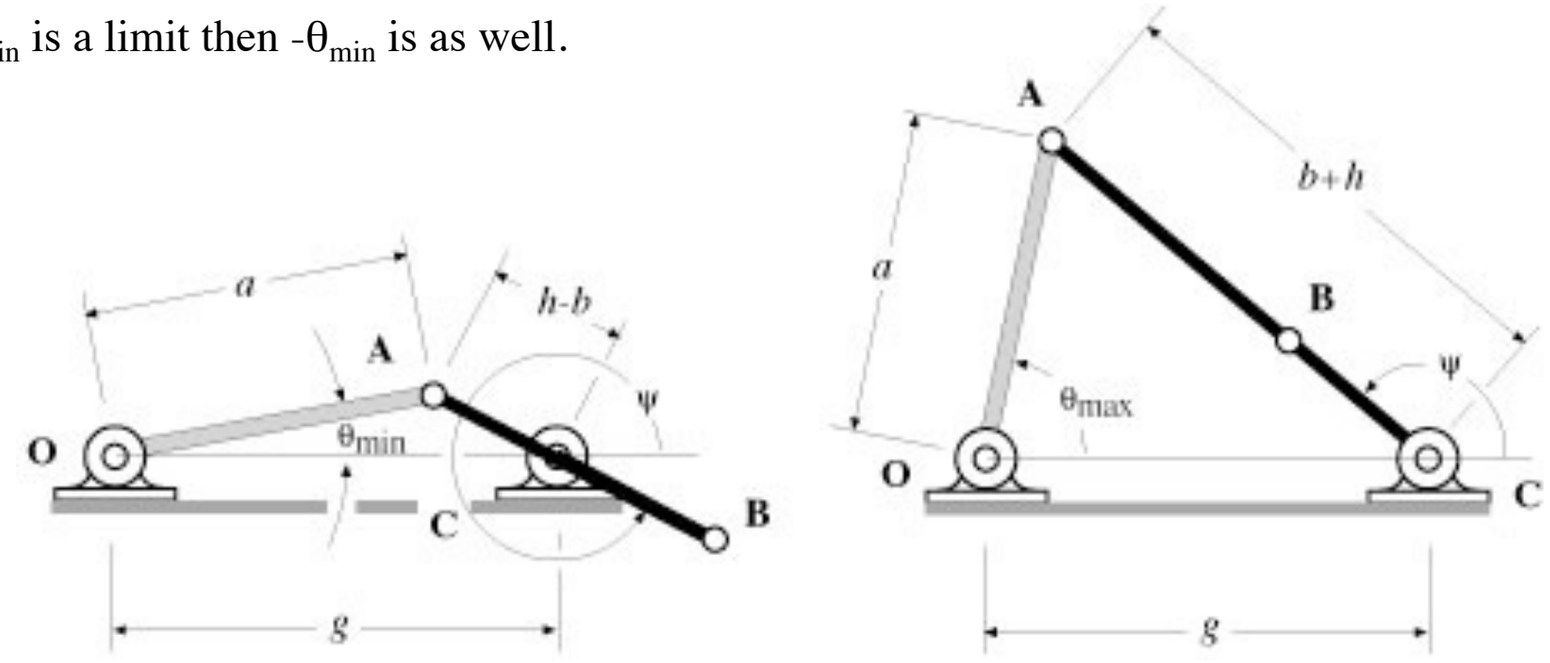


The Range of Crank Angles

• Limits to the input angle θ occur in the configurations shown in the figures below. Using the cosine law we obtain the two cases:

$\cos\theta_{min} = ((h-b)^2-a^2 - g^2)/2ag$, and $\cos\theta_{max} = ((h+b)^2-a^2 - g^2)/2ag$.

• It is possible that one, both or neither limit exists depending on the dimensions of the linkage. Also notice that arccos() has two values, therefore if θ_{min} is a limit then $-\theta_{min}$ is as well.



Thus, we have four cases:

1. Neither limit exists and the crank can make a full rotation;
2. only θ_{min} exists and the crank oscillates through the range $\theta_{min} \le \theta \le \pi$, and $-\pi \le \theta \le -\theta_{min}$;
3. only θ_{max} exists and the crank oscillates though the range $-\theta_{max} \le \theta \le \theta_{max}$; or
4. both limits exist and the crank oscillates in either range $\theta_{min} \le \theta \le \theta_{max}$ or $-\theta_{max} \le \theta \le \theta_{min}$.

Velocity Loop Equations

The loop equations of a four-bar linkage are give by

Position: $\boldsymbol{\gamma}\mathbf{Z}_1 + \boldsymbol{\varphi}\mathbf{Z}_2 = \mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3$.

The velocity loop equations are obtained by computing the time derivatives of this equation. Because vectors $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$ and $\mathbf{Z}_4$ are constant, we need only consider the derivative of a rotation operator $\boldsymbol{\theta}$ is given by

$$\dot{\boldsymbol{\theta}} = \frac{d}{dt}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = \dot{\theta}\begin{bmatrix}-\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta\end{bmatrix} = \omega_\theta \mathbf{J}\boldsymbol{\theta},$$

where ω_θ is the time derivative (angular velocity) of the angle θ. The matrix $\mathbf{J}$ rotates a vector by 90 degrees and has the property that $\mathbf{J}^2$=-1, where 1 is the 2x2 identity matrix.

It is convenient to use vector notation, by introducing the unit vector $\boldsymbol{k}$ perpendicular to the x-y plane, so we have $\mathbf{J}\mathbf{Z}_3 = \boldsymbol{k}\times\mathbf{Z}_3 = \mathbf{Z}_3^{\perp}$, so the vector $\mathbf{Z}_3^{\perp}$ is obtained from $\mathbf{Z}_3$ by a 90 degree rotation.

Therefore, we have

Velocity: $\omega_\gamma \boldsymbol{\gamma}\mathbf{Z}_1^{\perp} + \omega_\varphi\, \boldsymbol{\varphi}\mathbf{Z}_2^{\perp} = \omega_\eta \boldsymbol{\eta}\mathbf{Z}_3^{\perp}$,

where ω_γ, ω_φ, and ω_η are the angular velocities associated with the angles γ, φ, and η. Substitute $\mathbf{Z}_1 = \mathbf{A}\text{-}\mathbf{O}$, $\mathbf{Z}_2 = \mathbf{B}\text{-}\mathbf{A}$, $\mathbf{Z}_3 = \mathbf{B}\text{-}\mathbf{C}$, to obtain the equivalent version

$$\omega_\gamma \boldsymbol{\gamma}(\mathbf{A}\text{-}\mathbf{O})^{\perp} + \omega_\varphi \boldsymbol{\varphi}(\mathbf{B}\text{-}\mathbf{A})^{\perp} = \omega_\eta \boldsymbol{\eta}(\mathbf{B}\text{-}\mathbf{C})^{\perp}.$$

Given the input angular velocity ω_γ the velocity loop equations are solved to determine ω_φ, and ω_η.

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Velocity Pole (input side)

The movement of the coupler has the property that at every instant there is a point that has zero velocity, called the velocity pole.

Let the rotation of the **input crank** be parameterized by t, so we have $\gamma(t)$, then the coupler curve is given by

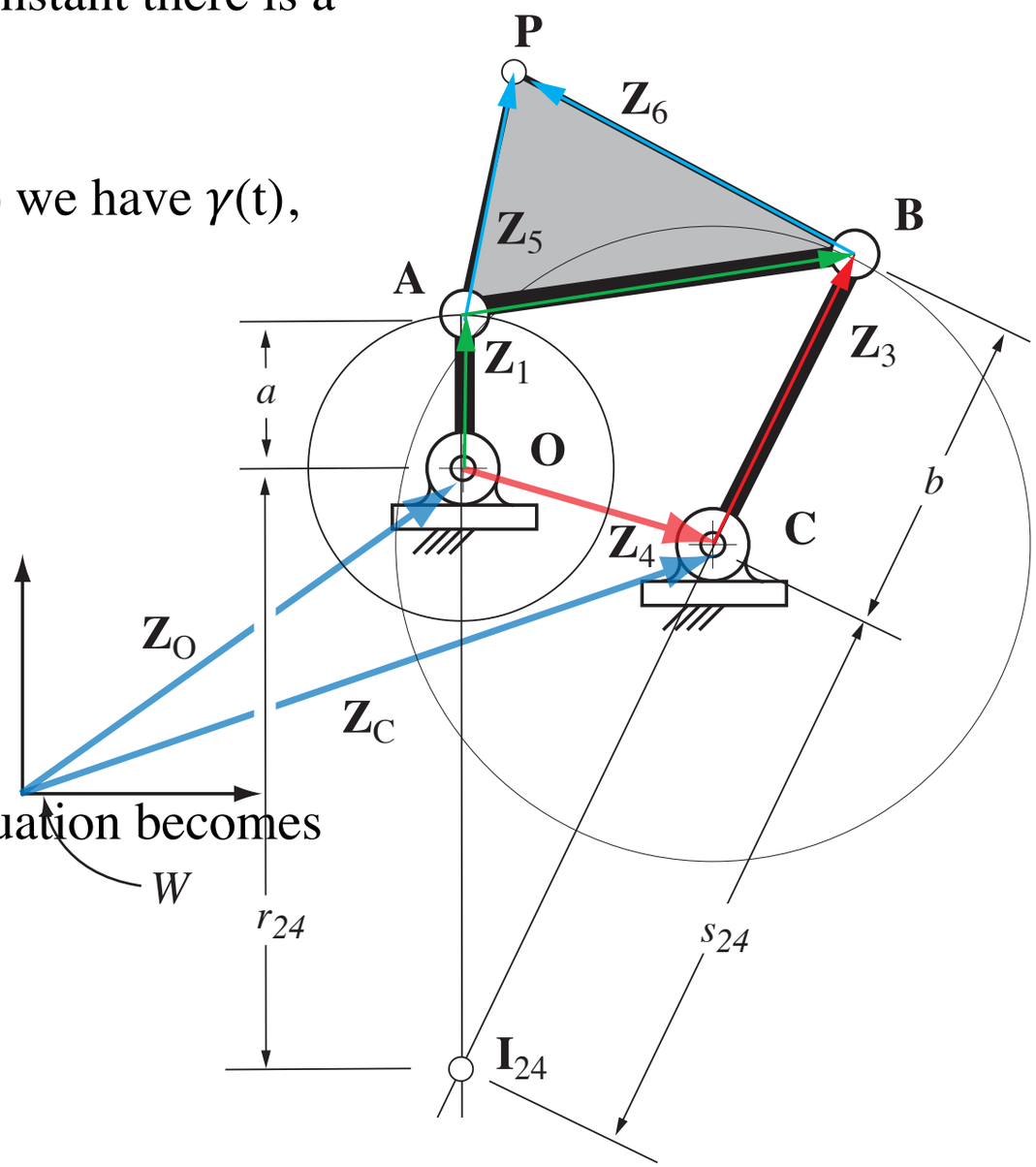
$$\mathbf{P}(\gamma(t)) = \mathbf{Z}_O + \boldsymbol{\gamma}(t)\mathbf{Z}_1 + \boldsymbol{\varphi}(t)\mathbf{Z}_5,$$

or

$$\mathbf{P}(t) = \mathbf{O} + \boldsymbol{\gamma}(t)(\mathbf{A}-\mathbf{O}) + \boldsymbol{\varphi}(t)(\mathbf{P}-\mathbf{A}),$$

The velocity of the coupler point is given by

$$\mathbf{V}_P = \omega_\gamma \boldsymbol{\gamma}(t)(\mathbf{A}-\mathbf{O})^\perp + \omega_\phi\, \boldsymbol{\varphi}(t)(\mathbf{P}-\mathbf{A})^\perp.$$



Let $\mathbf{A}_\gamma = \boldsymbol{\gamma}(t)(\mathbf{A}-\mathbf{O}) + \mathbf{O}$ and $\mathbf{P}_\varphi = \boldsymbol{\varphi}(t)(\mathbf{P}-\mathbf{A}) + \mathbf{A}_\gamma$, so this equation becomes

$$\mathbf{V}_P = \omega_\gamma(\mathbf{A}_\gamma-\mathbf{O})^\perp + \omega_\phi\,(\mathbf{P}_\varphi-\mathbf{A})^\perp.$$

There is a point **I** in the coupler that has *zero velocity*

$$\mathbf{V}_I = \omega_\gamma(\mathbf{A}_\gamma-\mathbf{O})^\perp + \omega_\phi(\mathbf{I}-\mathbf{A}_\gamma)^\perp = 0,$$

that is

$$\mathbf{I}_{24} = \mathbf{A}_\gamma - (\omega_\gamma/\omega_\phi)(\mathbf{A}_\gamma-\mathbf{O}).$$

This point, **velocity pole,** lies on the line through $\mathbf{A}_\gamma$ in the direction $\mathbf{A}_\gamma$-**O.**

Velocity Pole (output side)

The rotation of the **output crank** is also parameterized by t, so we have $\eta(t)$, then the coupler curve is given by

$$\mathbf{P}(\eta(t)) = \mathbf{Z}_C + \eta(t)\mathbf{Z}_3 + \varphi(t)\mathbf{Z}_5,$$

or

$$\mathbf{P}(t) = \mathbf{C} + \eta(t)(\mathbf{B}-\mathbf{C}) + \varphi(t)(\mathbf{P}-\mathbf{B}),$$

The velocity of the coupler point is now given by

$$\mathbf{V}_P = \omega_\eta \eta(t)(\mathbf{B}-\mathbf{C})^\perp + \omega_\phi \varphi(t)(\mathbf{P}-\mathbf{B})^\perp.$$

Introduce $\mathbf{B}_\eta = \eta(t)(\mathbf{B}-\mathbf{C}) + \mathbf{C}$ and $\mathbf{P}_\varphi = \varphi(t)(\mathbf{P}-\mathbf{B}) + \mathbf{B}_\eta$, so this equation becomes

$$\mathbf{V}_P = \omega_\eta(\mathbf{B}_\eta-\mathbf{C})^\perp + \omega_\phi(\mathbf{P}_\varphi-\mathbf{B}_\eta)^\perp$$

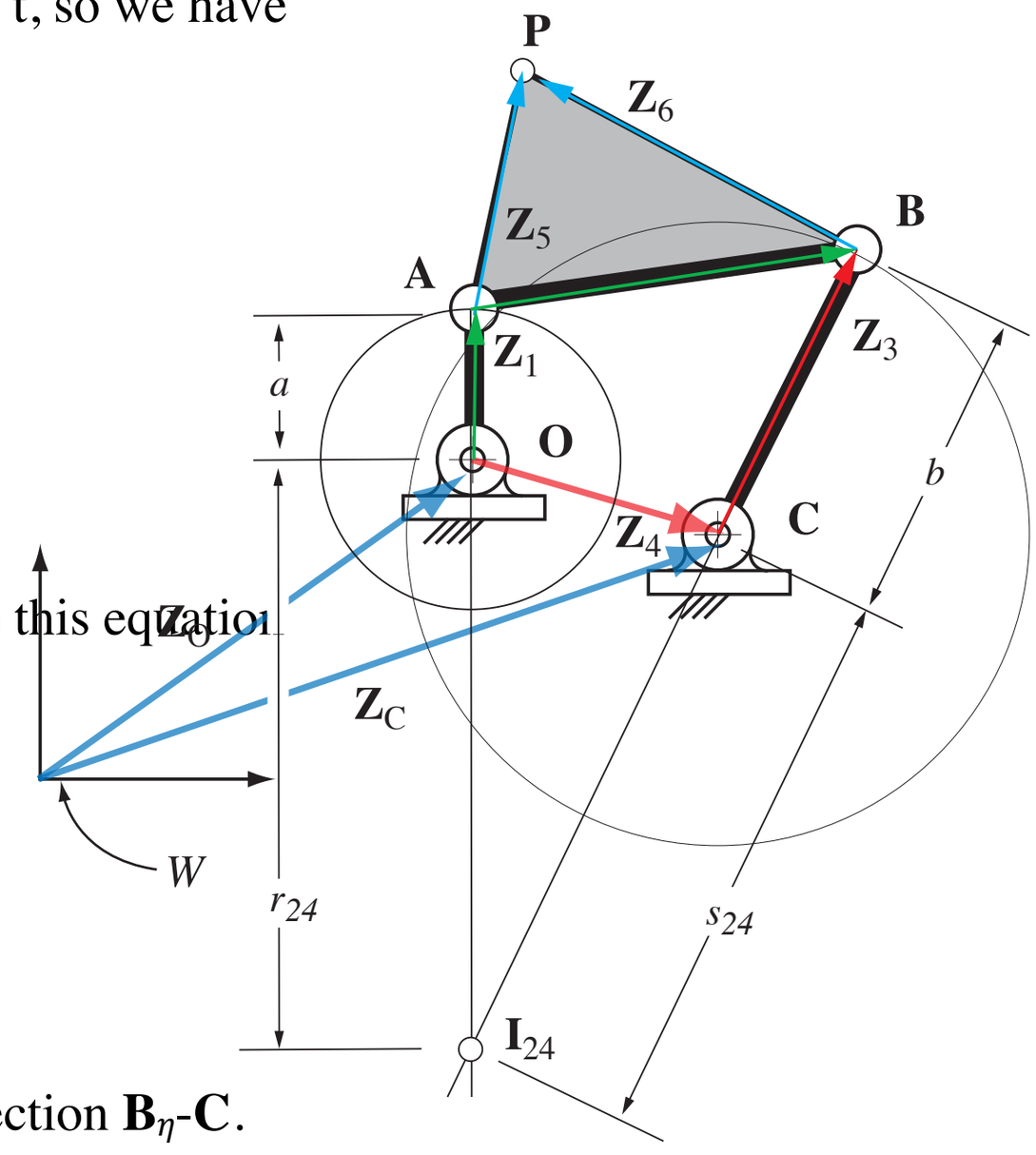
The coordinates of the velocity pole are now given by

$$\mathbf{V}_I = \omega_\eta(\mathbf{B}_\eta-\mathbf{C})^\perp + \omega_\phi(\mathbf{I}_{24}-\mathbf{B}_\eta)^\perp = 0,$$

so

$$\mathbf{I}_{24} = \mathbf{B}_\eta + (\omega_\eta/\omega_\phi)(\mathbf{B}_\eta-\mathbf{C}).$$

The velocity pole $\mathbf{I}_{24}$ lies on the line through $\mathbf{B}_\eta$ in the direction $\mathbf{B}_\eta$-$\mathbf{C}$.



The velocity pole lies on the intersection of the lines along the links $\mathbf{OA}_\gamma$ and $\mathbf{CB}_\eta$.

Velocity of Points in the Coupler

The velocity of points in the coupler can calculated in terms of their position relative to the velocity pole.

The velocity of the coupler point **P** is given by

$$\mathbf{V}_P = \omega_\gamma(\mathbf{A}_\gamma\text{-}\mathbf{O})^\perp + \omega_\phi\,(\mathbf{P}_\phi\text{-}\mathbf{A}_\gamma)^\perp,$$

and

$$\mathbf{V}_P = \omega_\eta(\mathbf{B}_\eta\text{-}\mathbf{C})^\perp + \omega_\phi(\mathbf{P}_\phi\text{-}\mathbf{B}_\eta)^\perp.$$

The definition of the velocity pole $\mathbf{I}_{24}$ is given by

$$\mathbf{V}_I = \omega_\gamma(\mathbf{A}_\gamma\text{-}\mathbf{O})^\perp + \omega_\phi\,(\mathbf{I}\text{-}\mathbf{A}_\gamma)^\perp = 0,$$

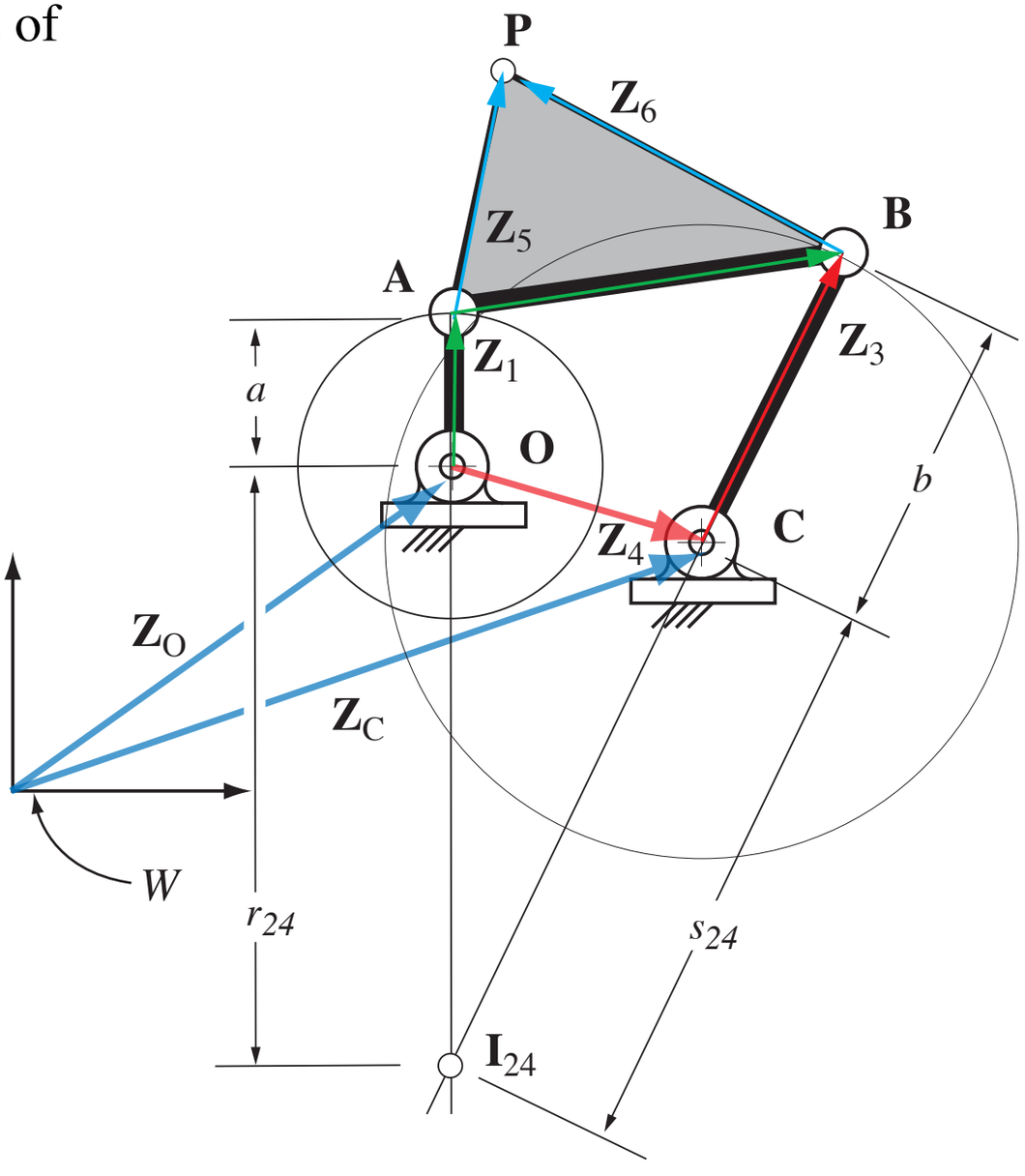
and

$$\mathbf{V}_I = \omega_\eta(\mathbf{B}_\eta\text{-}\mathbf{C})^\perp + \omega_\phi\,(\mathbf{I}_{24}\text{-}\mathbf{B}_\eta)^\perp = 0.$$

Compute $\mathbf{V}_P$ - $\mathbf{V}_I$ for both the input and output equations to obtain,

$$\mathbf{V}_P = \omega_\phi\,(\mathbf{P}_\phi\text{-}\mathbf{I}_{24})^\perp.$$

Thus, the velocity of points in the coupler can be computed directly from their position relative to the velocity pole $\mathbf{P}_\phi$-$\mathbf{I}_{24}$



Speed Ratio (reference position)

The closure equation of the linkage can be used to determine the input-output speed ratio, R= $\omega_\gamma/\omega_\eta$, by computing its derivative in the reference position,

$$h^2 = (\mathbf{Z}_4 + \mathbf{Z}_3 - \mathbf{Z}_1).(\mathbf{Z}_4 + \mathbf{Z}_3 - \mathbf{Z}_1),$$

that is

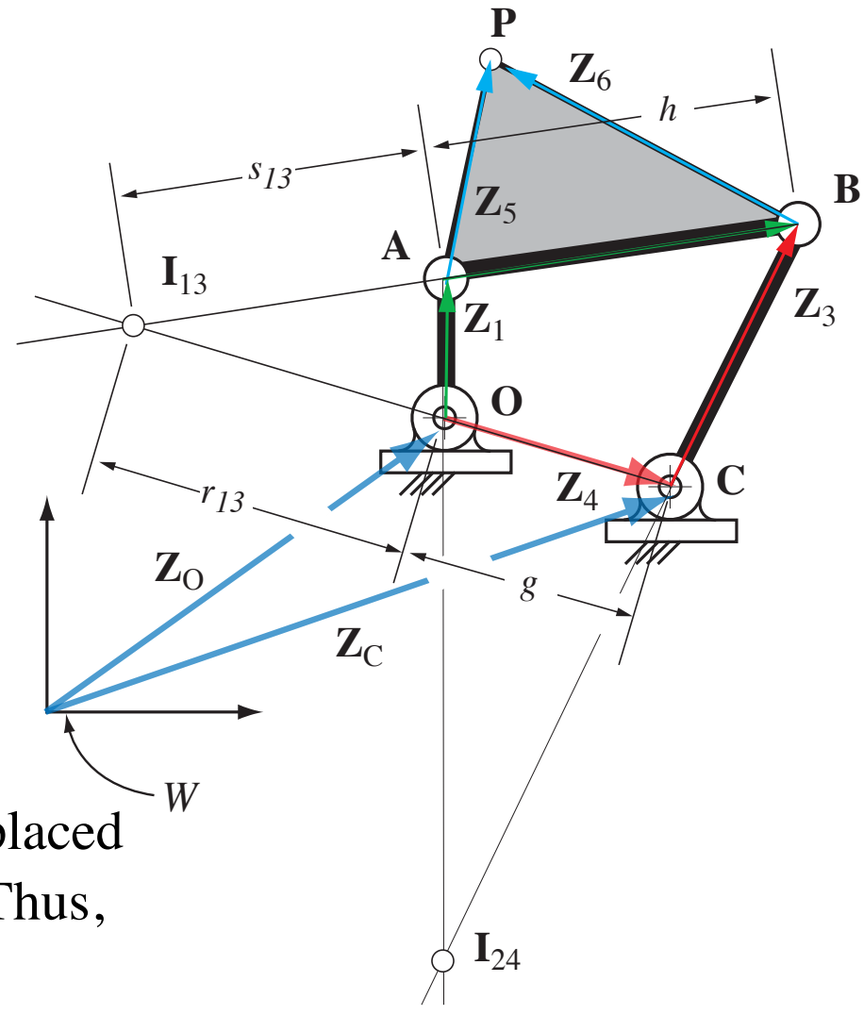
$$0 =(\omega_\eta \mathbf{Z}_3^\perp - \omega_\gamma \mathbf{Z}_1^\perp).\mathbf{Z}_2.$$

For convenience introduce the notation $\mathbf{Z}_3^\perp = \boldsymbol{k}\times\mathbf{Z}_3$, then we have

$$0 = (\omega_\eta\, \boldsymbol{k}\times(\mathbf{B}-\mathbf{C}) - \omega_\gamma\, \boldsymbol{k}\times(\mathbf{A}-\mathbf{O})).\ (\mathbf{B}-\mathbf{A})\ .$$

Factor out $\boldsymbol{k}\times$, then interchange the dot and cross products to obtain

$$\boldsymbol{k}.(\omega_\eta(\mathbf{B}-\mathbf{C}) - \omega_\gamma(\mathbf{A}-\mathbf{O})) \times (\mathbf{B}-\mathbf{A})= 0\ .$$



Notice that $\mathbf{B} \times (\mathbf{B}-\mathbf{A}) = \mathbf{A}\times (\mathbf{B}-\mathbf{A})$, which means **A** and **B** can be replaced by $\mathbf{I}_{13} = \mathbf{A} + t(\mathbf{B}-\mathbf{A})$, where $\mathbf{I}_{13}$ lies on the the line through **OC**. Thus, we have

$$\boldsymbol{k}.(\omega_\eta(\mathbf{I}_{13}-\mathbf{C}) - \omega_\gamma(\mathbf{I}_{13}-\mathbf{O})) \times (\mathbf{B}-\mathbf{A})=0.$$

Therefore, $\omega_\gamma/\omega_\eta = (r_{13}+g)/r_{13}$, where $r=|\mathbf{I}_{13}-\mathbf{O}|$ and $g=|\mathbf{I}_{13}-\mathbf{C}|$.

Speed Ratio (any position)

The closure equation in any configuration can be used to determine the associated input-output speed ratio, R= $\omega_\gamma/\omega_\eta$, by computing the derivative of,

$$h^2 = (\mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3 - \boldsymbol{\gamma}\mathbf{Z}_1).(\mathbf{Z}_4 + \boldsymbol{\eta}\mathbf{Z}_3 - \boldsymbol{\gamma}\mathbf{Z}_1),$$

that is

$$(\omega_\eta \boldsymbol{k}\times\boldsymbol{\eta}(\mathbf{B} - \mathbf{C}) - \omega_\gamma \boldsymbol{k}\times\boldsymbol{\gamma}(\mathbf{A} - \mathbf{O})).\boldsymbol{\varphi}(\mathbf{B} - \mathbf{A}) = 0,$$

where for convenience we have $[\mathrm{J}]\mathbf{Z}_3 = \boldsymbol{k}\times\mathbf{Z}_3$.

Introduce the notation $\mathbf{A}_\gamma = \boldsymbol{\gamma}(\mathbf{A}\text{-}\mathbf{O})) + \mathbf{O}$, and $\mathbf{B}_\eta = \boldsymbol{\eta}(\mathbf{B}\text{-}\mathbf{C}) + \mathbf{C}$, then $\boldsymbol{\varphi}(\mathbf{B}\text{-}\mathbf{A}) = \mathbf{B}_\eta - \mathbf{A}_\gamma$, so we have

$$\boldsymbol{k}\times(\omega_\eta(\mathbf{B}_\eta - \mathbf{C}) - \omega_\gamma(\mathbf{A}_\gamma - \mathbf{O})).(\mathbf{B}_\eta - \mathbf{A}_\gamma) = 0.$$

or

$$\boldsymbol{k}.(\omega_\eta(\mathbf{B}_\eta - \mathbf{C}) - \omega_\gamma(\mathbf{A}_\gamma - \mathbf{O})) \times (\mathbf{B}_\eta - \mathbf{A}_\gamma) = 0\ .$$

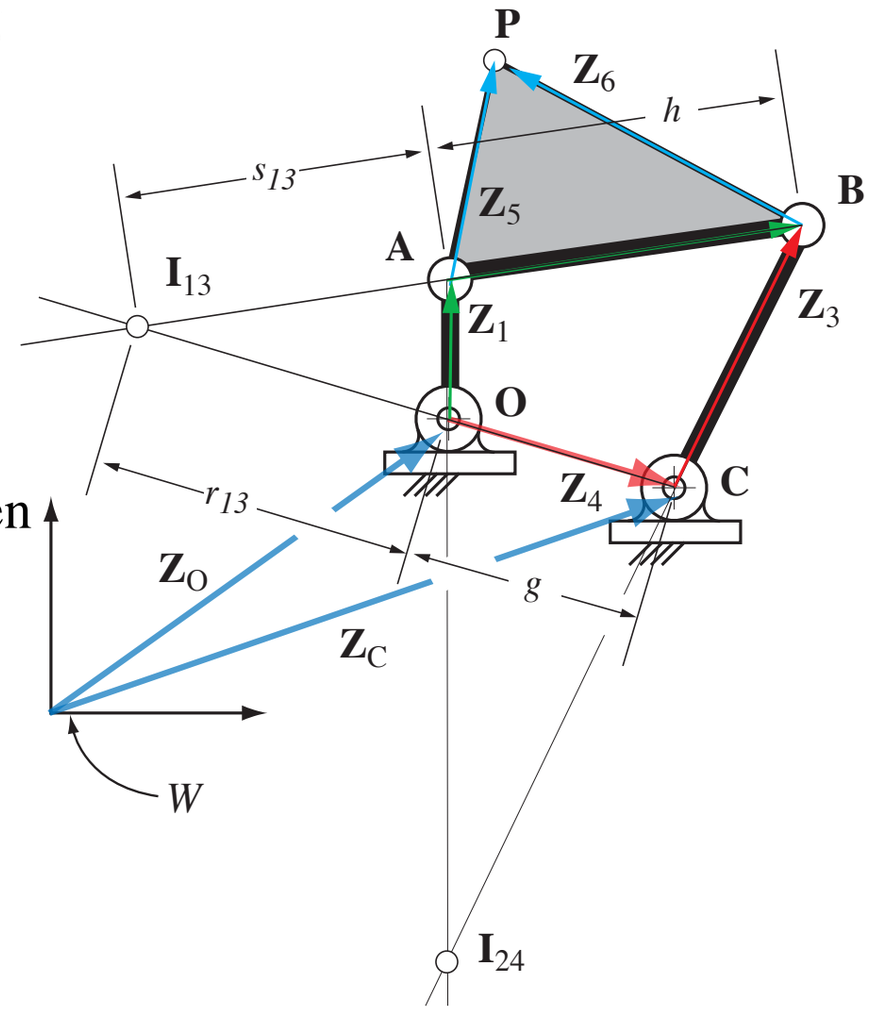
As we have seen before the points $\mathbf{A}_\gamma$ and $\mathbf{B}_\eta$ can be replaced by

$$\mathbf{I}_{13} = \mathbf{A}_\gamma + \mathrm{s}\,(\mathbf{B}_\eta - \mathbf{A}_\gamma) = \mathrm{r}_{13}\,\mathbf{e}_{OC} + \mathbf{O} = (\mathrm{r}_{13} + \mathrm{g})\,\mathbf{e}_{OC} + \mathbf{C}.$$

to define,

$$\boldsymbol{k}.(\omega_\eta(\mathbf{I}_{13} - \mathbf{C}) - \omega_\gamma(\mathbf{I}_{13} - \mathbf{O}))\times(\mathbf{B}_\eta - \mathbf{A}_\gamma) = 0$$

Therefore, $\omega_\gamma/\omega_\eta = (\mathrm{r}_{13}+\mathrm{g})/\mathrm{r}_{13}$.



Coordinates of the Instant Center

The coordinates of the instant center $\mathbf{I}_{13}$ can be computed directly from the condition that it is the intersection of lines along the links **OC** and **AB**.

Introduce the unit vectors $\mathbf{e}_{OC}= (\cos\theta_0, \sin\theta_0)$ and $\mathbf{e}_{AB}= (\cos\varphi_0, \sin\varphi_0)$ obtained by normalizing the vectors $\mathbf{Z}_4=\mathbf{C}-\mathbf{A}$ and $\mathbf{Z}_2=\mathbf{B}-\mathbf{A}$.

Then we solve for the distances s and t defined by the equation

$$\mathbf{I}_{13} = \mathbf{O} + r_{13}\,\mathbf{e}_{OC} = \mathbf{A} + s_{13}\,\mathbf{e}_{AB}.$$

Introduce $\mathbf{A}-\mathbf{O} = a\,\mathbf{e}_{OA} = a\,(\cos\gamma_0, \sin\gamma_0)$ and rearrange to obtain,

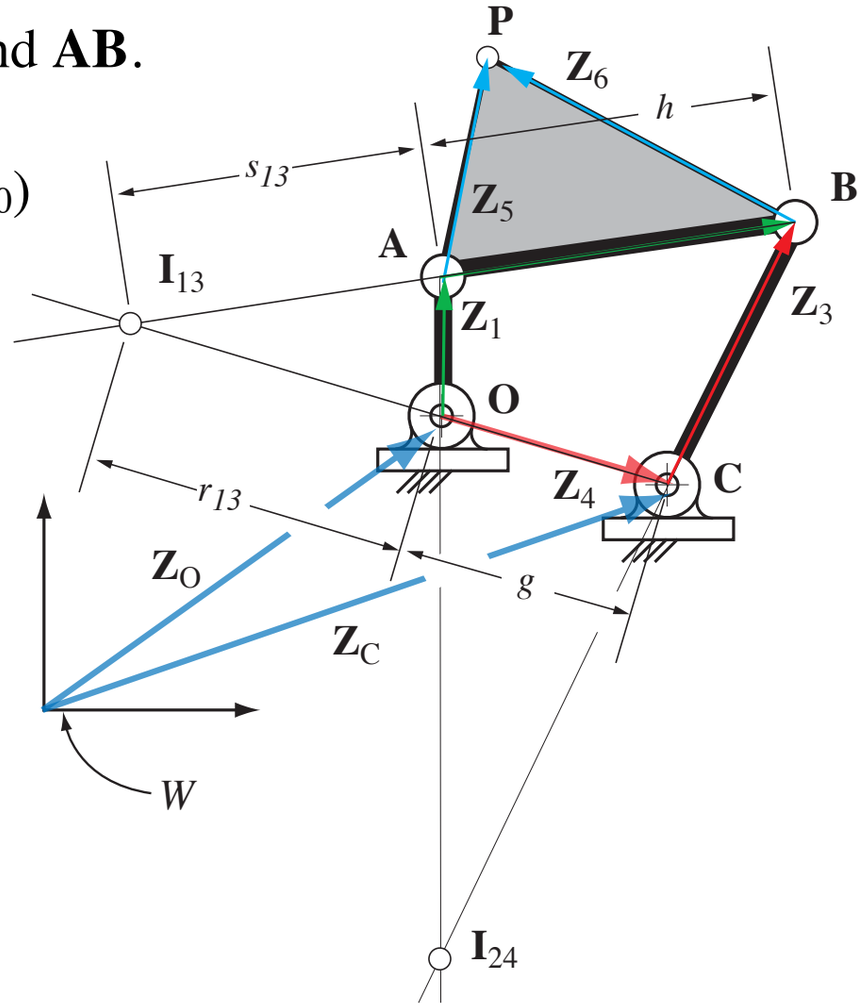
$$r_{13}\,\mathbf{e}_{OC} - s_{13}\,\mathbf{e}_{AB} = a\,\mathbf{e}_{OA}\ .$$

Solve these equations to obtain,

$$r_{13} = a\,\sin(\gamma_0 - \varphi_0) / \sin(\theta_0 - \varphi_0)$$

and

$$r_{13} = a\,\sin(\gamma_0 - \theta_0) / \sin(\theta_0 - \varphi_0).$$



Mechanical Advantage

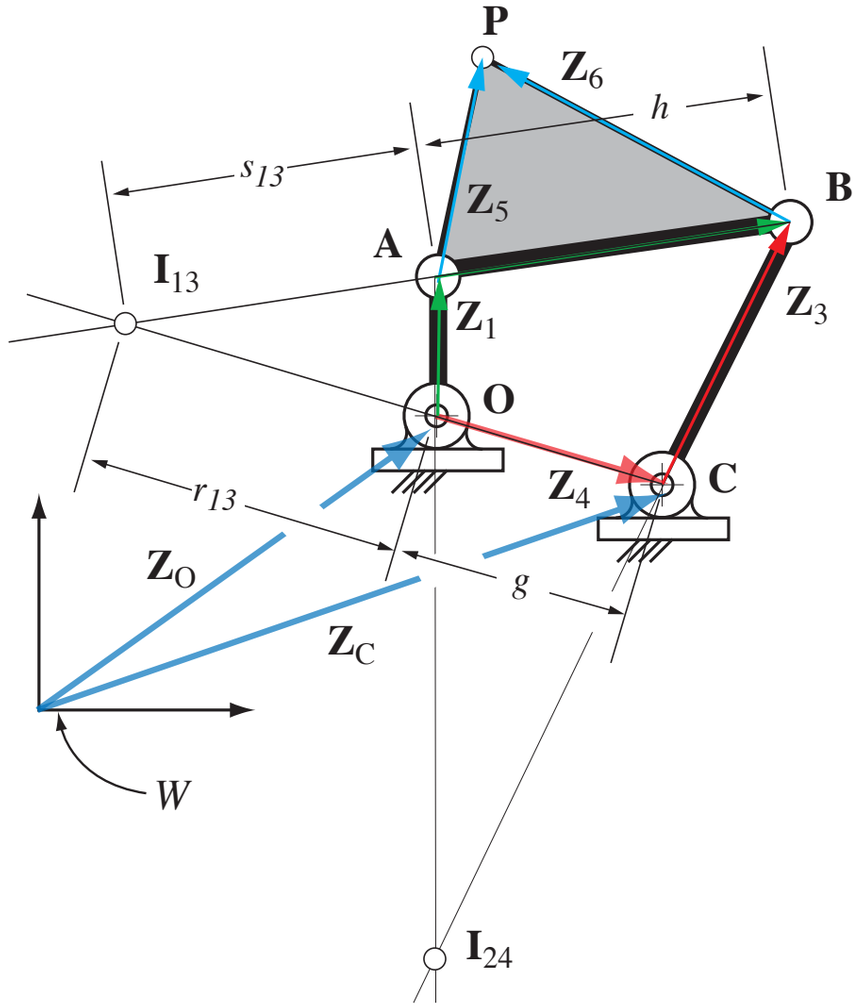
The principle of virtual work states that if there are no energy losses in a machine, known as an ideal machine, then the "power in equals power out." For a four-bar linkage the power in and power out are given by $P_{in}=T_{in}\omega_\gamma$ and $P_{out}=T_{out}\omega_\eta$.

The mechanical advantage of the linkage is the ratio of output torque to input torque, so we have

$$MA = T_{out}/T_{in} = \omega_\gamma/\omega_\eta.$$

Thus, for an ideal system the mechanical advantage is given by the speed ratio, $R=\omega_\gamma/\omega_\eta$, that is

$$MA = T_{out}/T_{in} = \omega_\gamma/\omega_\eta = (r_{13}+g)/r_{13}.$$



Summary

- An *ideal linkage* is a collection of rigid links connected by *ideal joints*. These assumptions allow the use of geometry to analyze the movement of the linkage and evaluate its mechanical advantage. Real linkage systems flex and lose energy through friction and wear.

- The *closure equation* of a linkage is defined by the requirement that input and output links must move in a way that maintains a constant length for the coupler link. Differentiation of this constraint yields the input-output speed ratio and mechanical advantage.

- The *position loop equations* are used to compute the coupler angle.

- The first derivative of the loop equations define the velocity loop equations, which are used to compute the angular velocities of the output crank and coupler link.

- The ratio of the input to output angular velocities of the linkage define its mechanical advantage.