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Type Synthesis

Gruebler's Equation, Assur Groups, Baranov Trusses, Graph Theory and Rigidity Theory

Simple Machines

- The classification of machines was introduced during the Renaissance and was based on Hero of Alexandria's Mechanics which identified five lift mechanisms (Usher, Abbott Payson (1988). A History of Mechanical Inventions. USA: Courier Dover Publications.):

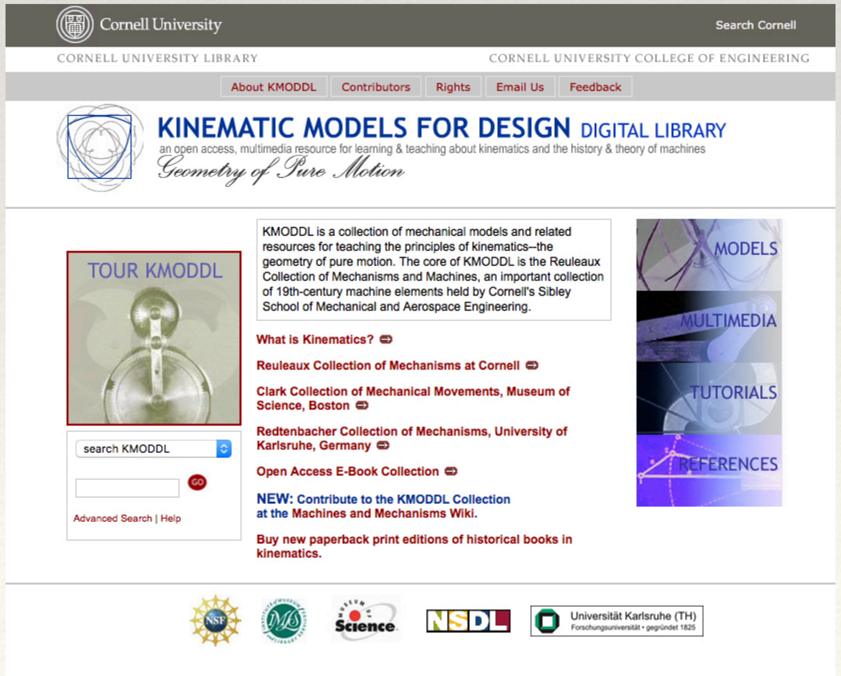
  the lever, windlass, pulley, wedge and screw.

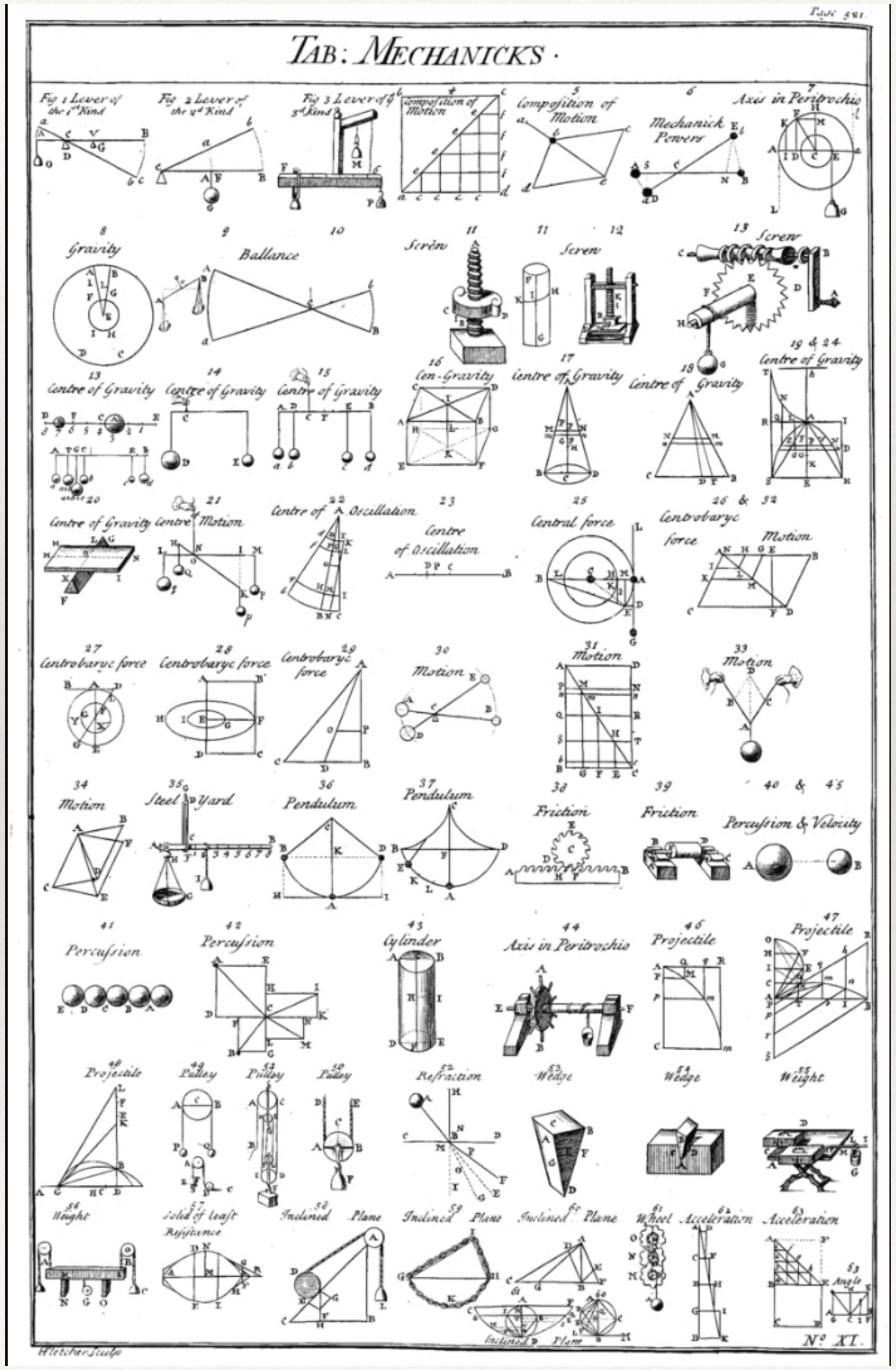
- These five simple machines are taught in elementary school, with the idea that more complicated machines, often called compound machines, are constructed from these elements.

Machine Catalogs

- The inclined plan was added to this list during the Renaissance, and since that time catalogs of special machines were generated through the 1800's. (The figure shows a page from Chambers' Cyclopaedia published in 1728.)

- Franz Reuleaux assembled over 800 models that represent elementary machines. He defined machines as assemblies of links and joints. (F. Realeaux, Lehrbuch der Kinematik, Vol 1, 11, Braunschweig: F. Vieweg und sohn)





Gruebler's Equation

- Abstraction of a machine as an assembly of links and joints provides a formula for its degrees of freedom, or mobility, known as Gruebler's equation:

$$\text{links}: \quad N = n_2 + n_3 + n_4 + \ldots = \sum_{p=2} n_p,$$

$$\text{joints}: \quad 2J = 2n_2 + 3n_3 + 4n_4 + \ldots = \sum_{p=2} p n_p,$$

$$\text{mobility}: \quad F = 3(N-1) - 2J.$$

- Crossley used this formula to identify one degree of freedom linkages by setting F=1 and solving to obtain:

$$J = \frac{3}{2}N - 2$$

- He used this to identify that one degree of freedom systems must have an even number of links,

- And that two ternary links can be added to any system of N links without changing the degrees of freedom,

Table 1 $X = 1$ mechanisms

Group	N	n_2	n_3	n_4	n_5	n_6
I	4	4	..	..	..	..
II	6	4	2	..	..	..
		5	0	1	..	..
III	8	4	4	..	..	..
		5	2	1	..	..
		6	0	2	..	..
		6	1	0	1	..
IV	10	4	6	..	..	..
		5	4	1	..	..
		6	3	0	1	..
		6	2	2	..	..
		7	1	1	1	..
		7	0	3	0	..
		8	0	0	2	..
		8	0	1	0	1
V	12	4	8	..	..	..
		5	6	1	..	..
		6	4	2	..	..
			etc.			

F. R. E. Crossley, Contribution to Gruebler's Theory in the Number Synthesis of Plane Mechanisms, J. Eng. Ind 86(1), 1-5 (Feb 01, 1964)

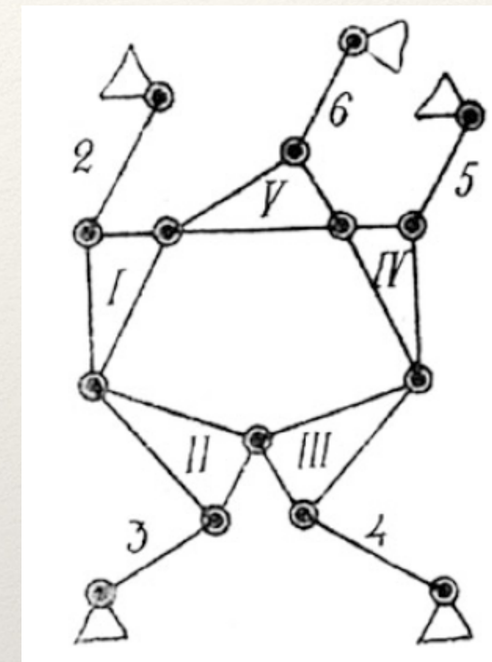
Assur Groups

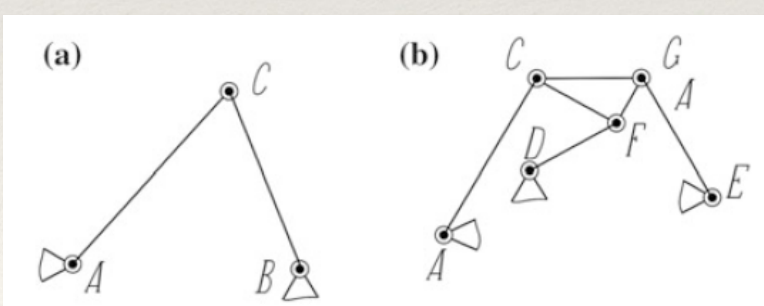
- Leonid Assur (1878-1920) noticed that systems of links that satisfy the equation

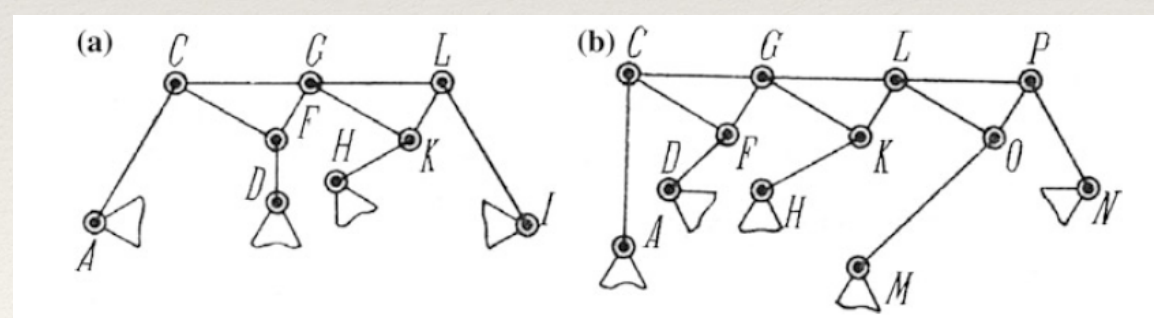
$$3N - 2J = 0$$

will have zero degrees of freedom when attached to a fixed ground link. It is called an Assur Group.

- In fact, an Assur group can be added to any mechanism without changing its degrees of freedom.
- Wohlhart states that a class 3 order 6 Assur group has the greatest closed polygon with three sides and six external joints for input. Therefore, the closed chain shown is a class 5, order 5 Assur Group. (K. Wohlhart, Position analyses of normal quadrilateral Assur groups, Mechanism and Machine Theory 45 (2010) 1367–1384)
- When the external joints of an Assur Group are attach to a single ground link, the system becomes a statically determinate Baranov Truss.







A. Evgrafov and D. Kozlikin, Leonid Assur (1878-1920), M. Ceccarelli (ed.), Distinguished Figures in Mechanism and Machine Science History of Mechanism and Machine Science 26, DOI: 10.1007/978-94-017-8947-9_2, Springer Science+Business Media Dordrecht 2014

Baranov Truss

- A Baranov truss is a statically determinate truss, also a mechanism with zero degrees of freedom, so

$$F = 3N - 2J - 3 = 0 \quad \text{or} \quad J = \frac{3}{2}(N-1)$$

- This shows that a Baranov truss must have a odd number of bodies.
- A given Baranov truss can be transformed into an Assur group by removing any one link.
- The joints exposed by removing the link become the external joints available for attachment to a mechanism.

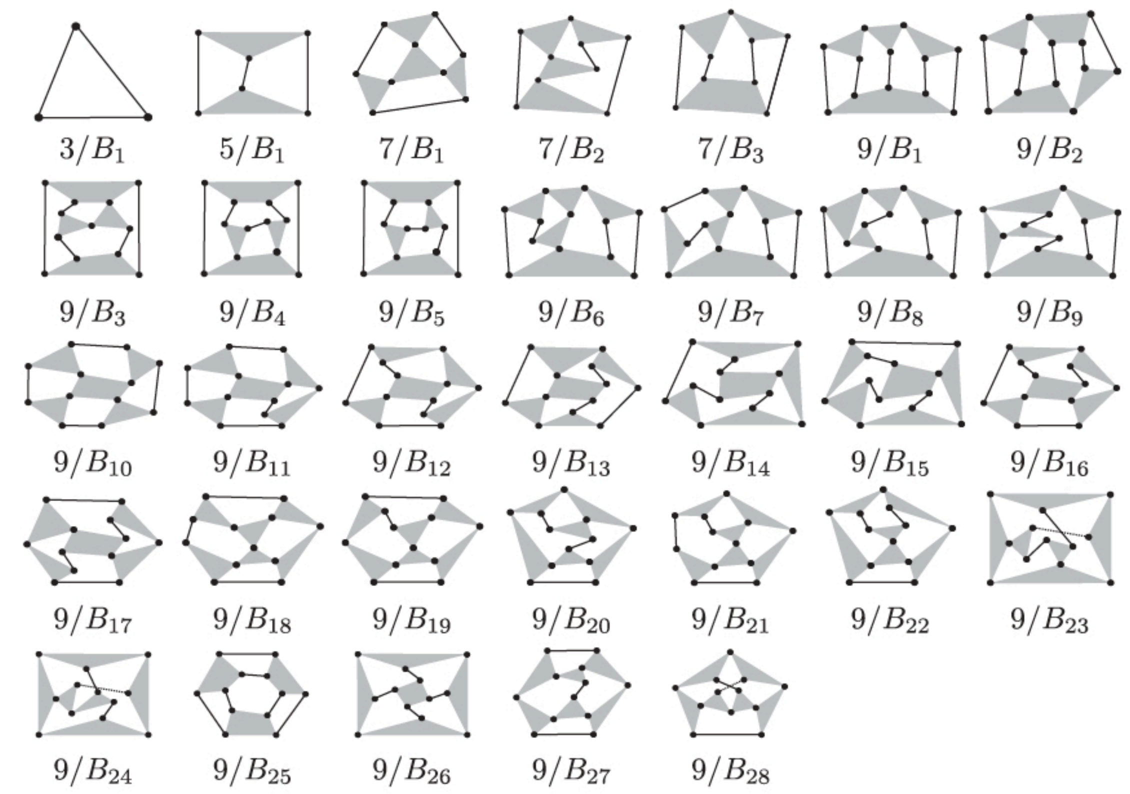
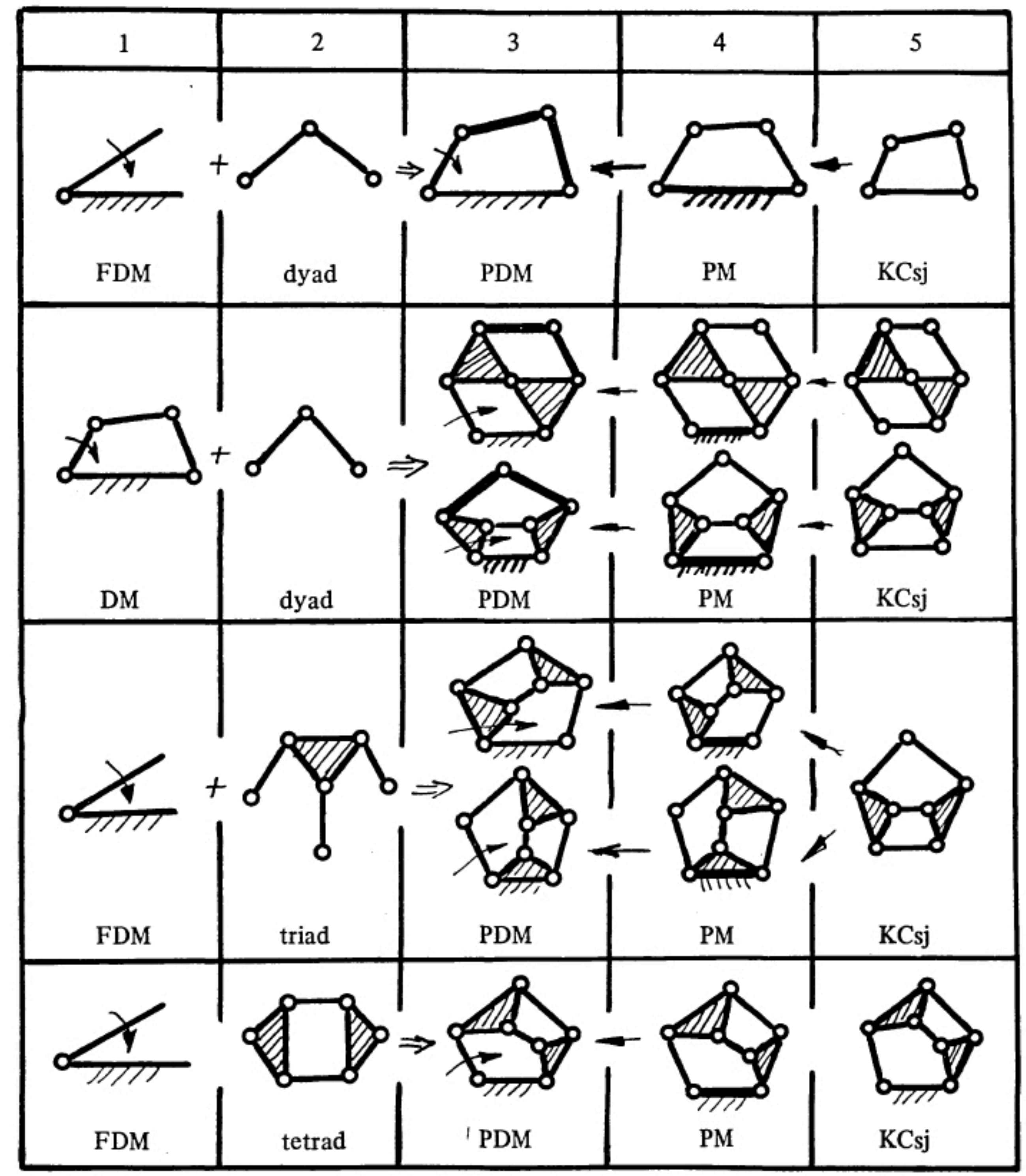


Fig. 1. The cataloged Baranov trusses.

N. Rojas and F. Thomas, On closed-form solutions to the position analysis of Baranov trusses, Mechanism and Machine Theory 50 (2012) 179–196

Assembling Linkages

- Manolescu illustrates how a driving mechanism (DM) and Assur group are combined to form a new mechanism.
- He also shows how a linkage topology is specialized by selecting ground and input links.



N. I. Manolescu, A unified method for the formation of all planar jointed kinematic chains and Baranov trusses, Environment and Planning B, 1979, volume 6, pages and 447-454

Graph Theory

- A graph can be associated with a mechanism by identifying the links as vertices and the joints as edges.
- A contracted graph collects a serial chain of joints into a single edge.

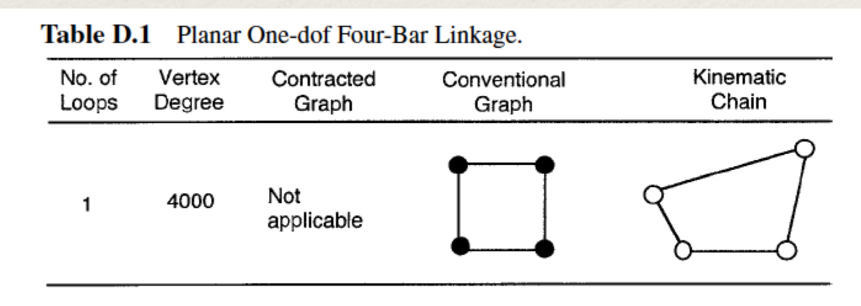


Table D.1 Planar One-dof Four-Bar Linkage.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
1	4000	Not applicable		

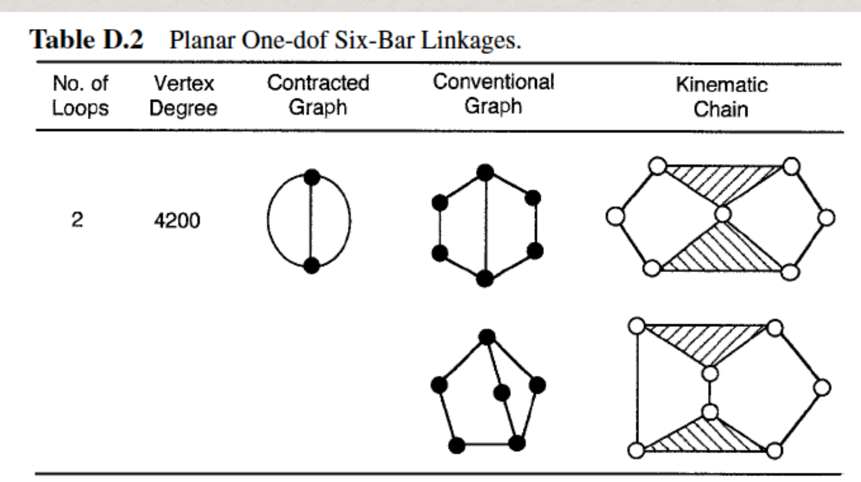


Table D.2 Planar One-dof Six-Bar Linkages.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
2	4200			

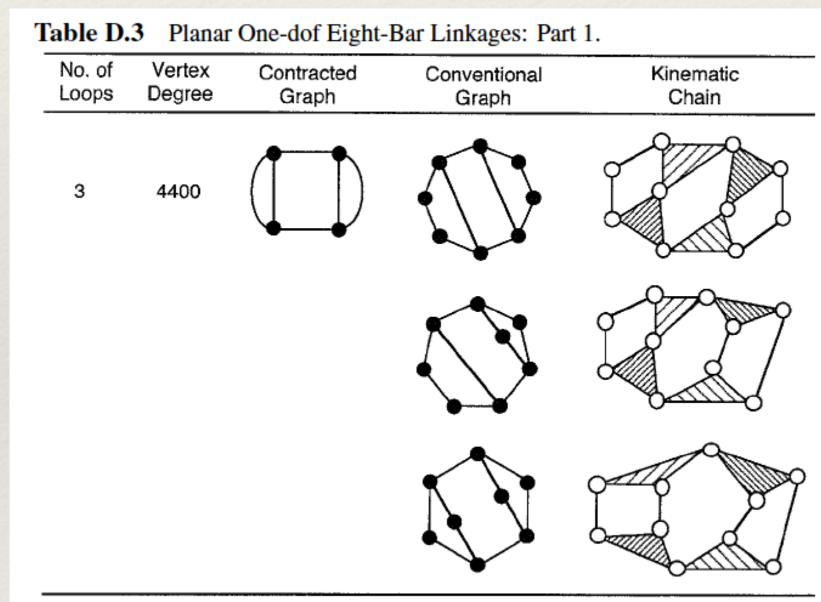


Table D.3 Planar One-dof Eight-Bar Linkages: Part 1.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
3	4400			

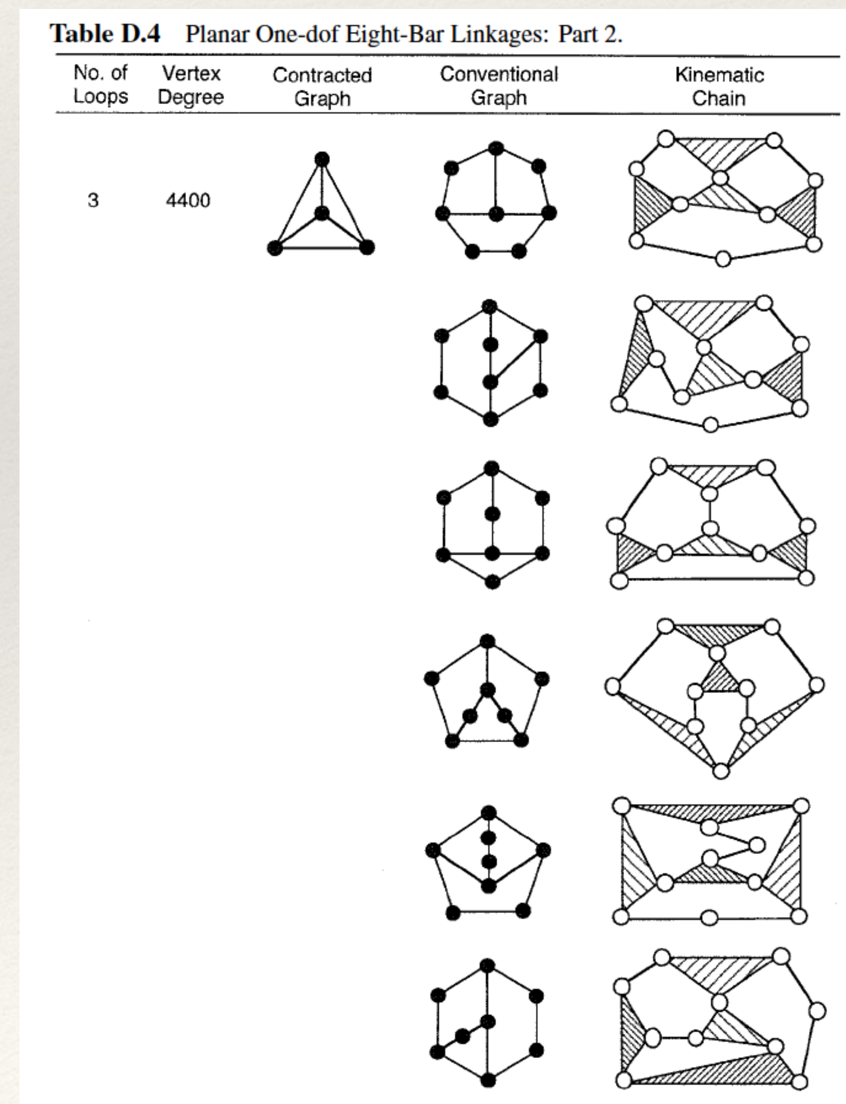


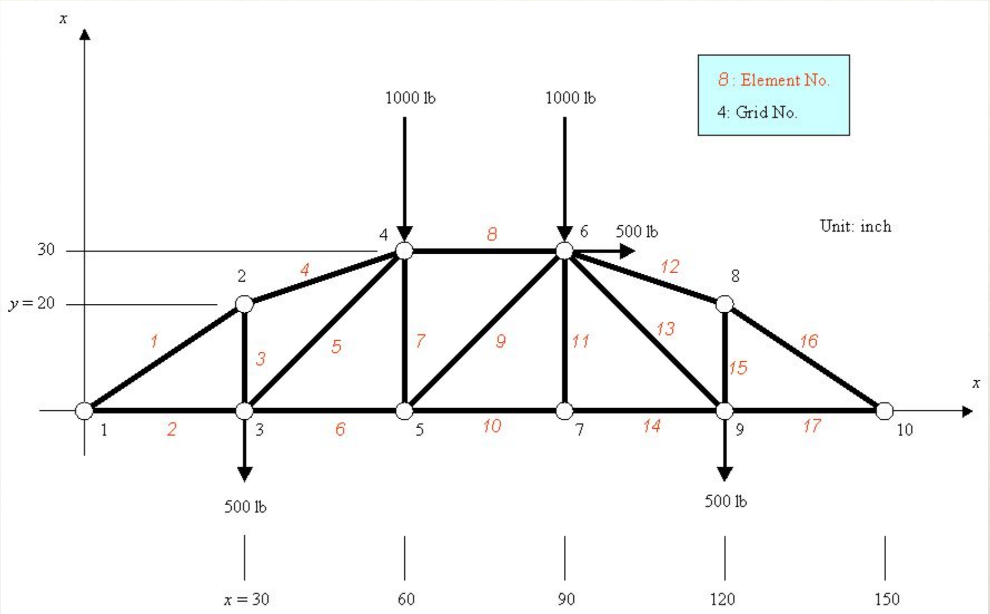
Table D.4 Planar One-dof Eight-Bar Linkages: Part 2.

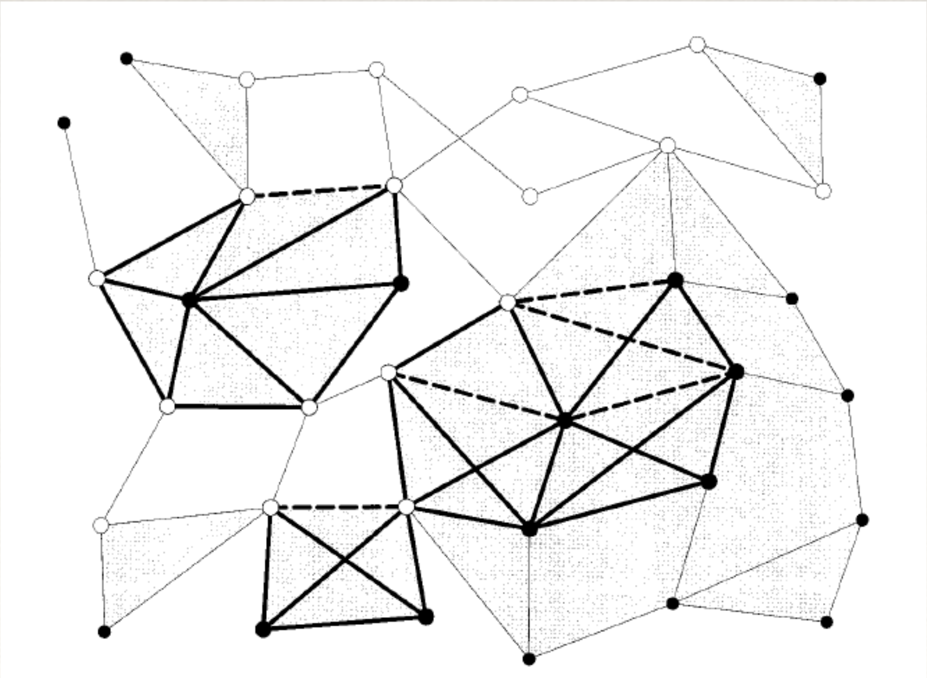
No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
3	4400			

L. W. Tsai, Mechanism Design: Enumeration of Kinematic Structures According to Function, CRC Press, Boca Raton, 2000.

Rigidity Theory

- Civil engineers define a truss to be an assembly of two force members connected by pin joints.
- For mechanical engineers this requires all polygons to be triangulated. Thus, the truss becomes a graph with joints called vertices, v, and binary links called edges, e.
- A truss is statically determinate, or isostatic, if the forces along each of the edges can be determined using the equilibrium equations at each joint.
- There are 2v joint equilibrium equations and e link forces and 3 ground reaction forces, so 2v = e+3, or 2v-3-e=0.
- A graph is defined to be *minimally rigid* if e=2v-3, and all subgraphs have at least e'=2v'-3. Minimally rigid is the same as statically determinate and isostatic. (Laman Theorem)



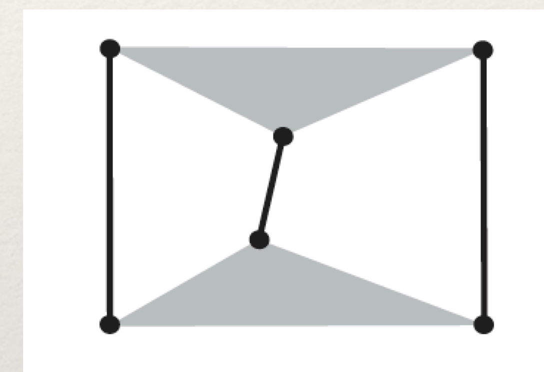


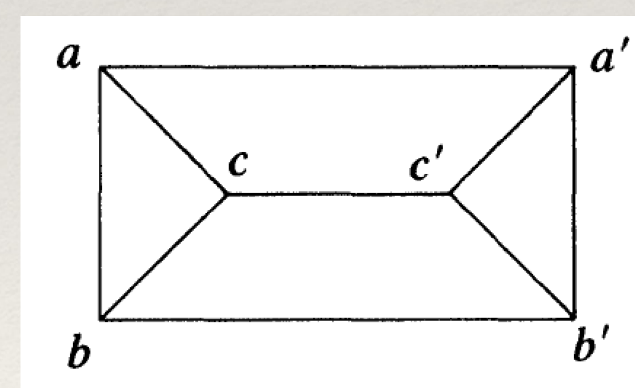
Baranov Graphs

- We can revise the definition of Baranov truss to reduce the polygons to two force members, or binary links.
- A ternary link yields 3 binary links so $e_3 = 3n_3$, a quaternary link becomes two triangles with a shared edge, or $e_4=(3.2\text{-}1)n_4$, a p-gon becomes $e_p=(3.(p\text{-}2)\text{-}(p\text{-}3))n_p$, $e_p=(2p\text{-}3)n_p$. Thus, the rigid links become minimally rigid subgraphs.
- Compute the mobility of the Baranov graph constructed from binary links. Let the number of binary links that meet at the vertex v be defined as its valence k. In Gruebler's equations joints that have a valence k are counted as k-1 joints.
- Let e be the number of links and J_k be the number of vertices of valence k, then Gruebler's formula becomes

$$\begin{aligned} F =& 3(e-1) - 2\sum_{k=2}(k-1)J_k = 0, \\ =& 3e - 3 - 2\sum_{k=2} kJ_k + 2\sum_{k=2} J_k = 0, \\ =& 3e - 3 - 2(2e) + 2v = 0, \\ =& 2v - e - 3 = 0. \end{aligned}$$

- Thus, Gruebler's formula is equivalent to the condition for minimal rigidity for graphs.





Rigidity Matrix

- If a graph with v vertices are assigned coordinates $\mathbf{p}_i$, i=1,…,v, in the plane, then it is called a *framework*
- The rigidity of a framework is studied using the e length equations $(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{p}_i - \mathbf{p}_j) = L_{ij}.^2$
- The framework moves then the velocities of each vertex must satisfy the equation $(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{v}_i - \mathbf{v}_j) = 0$, where $\mathbf{v}_i$ is the velocity of $\mathbf{p}_i$. This yields the e by 2v matrix equation, [R]m=0, where R is called the Rigidity matrix.
- For example a graph with four vertices and six edges has the rigidity matrix

$$\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = (p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{41}, p_{42});$$

$$\begin{bmatrix}
p_{11}-p_{21} & p_{12}-p_{22} & p_{21}-p_{11} & p_{22}-p_{12} & 0 & 0 & 0 & 0 \\
p_{11}-p_{31} & p_{12}-p_{32} & 0 & 0 & p_{31}-p_{11} & p_{32}-p_{12} & 0 & 0 \\
p_{11}-p_{41} & p_{12}-p_{42} & 0 & 0 & 0 & 0 & p_{41}-p_{11} & p_{42}-p_{12} \\
0 & 0 & p_{21}-p_{31} & p_{22}-p_{32} & p_{31}-p_{21} & p_{32}-p_{22} & 0 & 0 \\
0 & 0 & p_{21}-p_{41} & p_{22}-p_{42} & 0 & 0 & p_{41}-p_{21} & p_{42}-p_{22} \\
0 & 0 & 0 & 0 & p_{31}-p_{41} & p_{32}-p_{42} & p_{41}-p_{31} & p_{42}-p_{32}
\end{bmatrix}$$

- A framework is rigid if the range of the rigidity matrix is rank(R)=2v-3.

Henneberg Method

- Henneberg constructions add vertices and edges without changing rigidity.

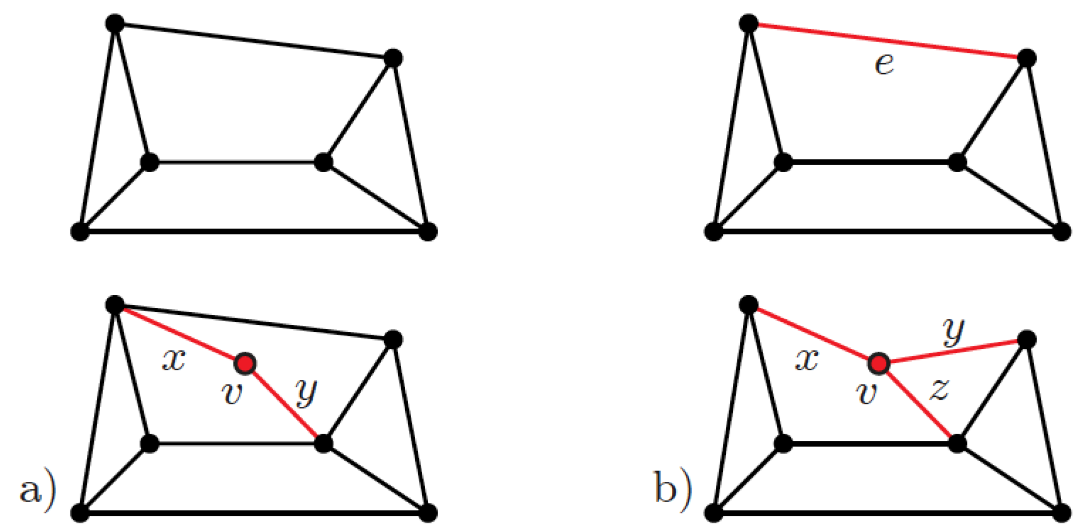


Figure 5: Building up an independent set of edges by: 2-valent vertex (a) and edge-split (b)

- Removing one bar transforms a statistically determinate structure into a one degree of freedom mechanism.

Pebble Game

- Linear dependence of the rows of the rigidity matrix defines a *matroid*.
- The evaluation of the rigidity of an arbitrary graph can be calculated using an algorithm known as the Pebble Game.
- D. J. Jacobs and B. Hendrickson, An Algorithm for Two-Dimensional Rigidity Percolation: The Pebble Game, JOURNAL OF COMPUTATIONAL PHYSICS 137, 346–365 (1997)

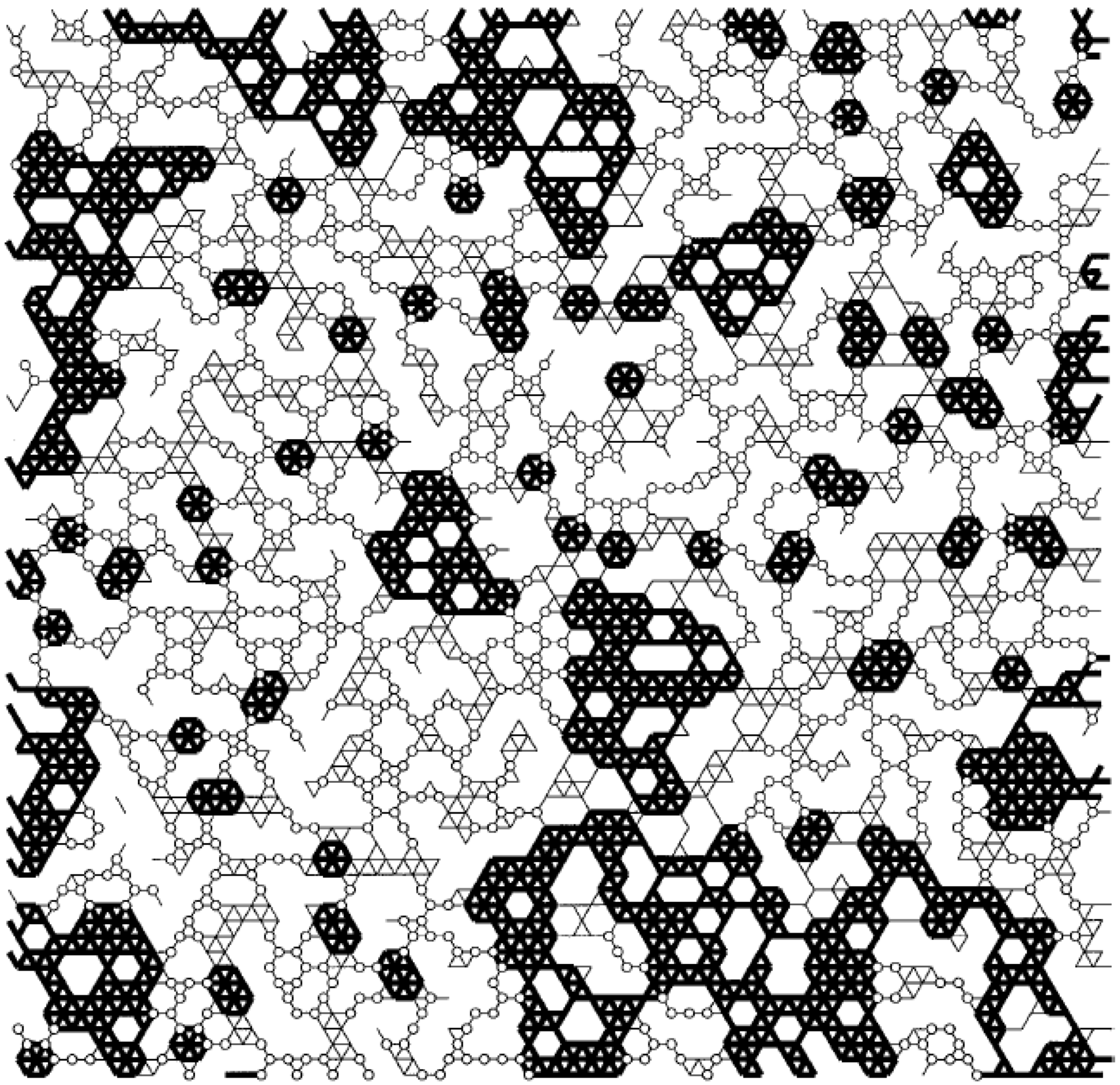


FIG. 7. The *topology* of a section of a site-diluted generic triangular network below the percolation threshold ($q = 0.6825$). Any particular realization will have distortions (not shown). The thin (thick) lines denote isostatic (overconstrained) bonds. The open circles denote pivot sites.